Conceptual Metaphors and Tacit Models in the Study of Mathematical Infinity

https://doi.org/10.3991/ijet.v17i15.33271

Tamara Díaz-Chang[1,2](✉), Elizabeth-H Arredondo[2]
[1] Universidad Austral de Chile, Los Ríos, Chile
[2] Universidad de Los Lagos, Osorno, Chile
tamara.diaz@uach.cl

Abstract—This work shows the connections between *conceptual metaphors* and unconscious, *tacit models* that benefit our understanding of mathematical infinity in the university classroom. From the perspective of cognitive linguistics, it is argued that *conceptual metaphors* play a key role in explaining how this mathematical concept is grounded in our experience, simultaneously providing a mechanism to address these *tacit models* in a more conscious way. Moreover, it is shown that *conceptual metaphors* can be built from the conflicting cognitive structures underlying these models, specifying obstacles and difficulties that teachers must consider when designing activities aimed at achieving an adequate understanding of mathematical infinity. This type of study allows us to improve our teaching practice, making us aware and stimulating students to become aware, to reflect on the inconsistencies of their own thoughts and intuitions regarding this mathematical concept. At the same time, it allows us to show the validity of these inconsistencies by revealing how our cognitive processes are constrained by bodily-grounded experiences determined by the complexity of our human nervous system. The use of technology would also engage students in these reflections and could also help them by fostering new ways of thinking about mathematical infinity. This perspective would be of interest in the current context of digital technology in mathematics education research.

Keywords—cognitive linguistics, higher education, basic mapping of infinity

1 Introduction

Mathematical infinity is one of the most complex and counterintuitive concepts faced by students and teachers in the study of mathematics at the university level. Full of paradoxes and controversies, it has played a central role in defining many areas of mathematics. For this reason, its learning and teaching processes have been widely studied from different theoretical perspectives in mathematics education. Despite this, it is not possible to affirm that we understand the complicated cognitive processes that develop in relation to its understanding.

There are numerous works (e.g., [1], [2], [3]) that account for the difficulty and complexity of epistemological obstacles present in this case. Furthermore, the classic philosophical debate on infinity in the actual and the potential sense originated in ancient Greece by Aristotle (as stated in [4]), has inspired different investigations (e.g., [5], [6], [7], [8]). For example, Arrigo and D'Amore [9] argue that the evolution of mathematical infinity in the actual sense occurs slowly, in a contradictory way, after a long process of maturation and cognitive systematization.

Usually, for students, the size of the set of integers is greater than the size of the set of natural numbers, some even say that it is the double. However, once the notion of same size (cardinality) of two sets with infinitely many elements is accepted, many students then believe that they have been able to conclude this because both sets have infinitely many elements, so they infer that all infinite sets have the same cardinality: infinity. In this case students conclude that the sets of natural numbers, the set of integers, the set of rational numbers and the set of real numbers should have the same cardinality or size. So here again a difficulty arises when dealing with different sizes of infinite sets introduced by Cantor, and the notion that there are infinities larger than others. Classic studies carried out in the 90s give evidence of this difficulty [7], [10], [11], which is called the *squeezing* of transfinite cardinals [12]. Previously, in one of his works, Duval [13] also analyzes the difficulty that students have in accepting the one-to-one correspondence between the set of natural numbers and its subset of square numbers, arguing that this is due to an obstacle that he calls *slipping*. Also, in relation to the above, the difficulty known in the literature as *inclusion* is reported and it is explicitly shown by the Galileo's paradox [2].

In some works, it is considered that intuitions about infinity emerge in consideration of recursive processes and as an extrapolation of our finite experience [9]. Children conceive potential infinity through the endless process of counting natural numbers. Later, when the symbol $\mathbb{N}$ for the set of natural numbers is introduced, they assume that the totality of these numbers can be considered. According to a study by Tall [14], most university students lose the conception of potential infinity in relation to natural numbers after the study of Set Theory, adopting the notion of actual infinity introduced by Cantor.

Another widespread intuitive conviction among students is to think that there are more points in a longer segment than in a shorter one [14], [15] which D'Amore [12] calls *dependency* of transfinite cardinals to properties related to the geometric measure of segments. The results of several studies [2], [16] show that many of the well-known paradoxes about infinity (such as the paradox of Zeno on Achilles and the Tortoise [17]) emerge as intuitive convictions that cause difficulties in the process of emergence of the meanings of this mathematical concept in students.

Fishbein [2] claims that *dependency* as well as *inclusion*, *squeezing*, *slipping* and other similar types of intuitive convictions appear when we deal with concepts that are too abstract or complex. In these cases, we have a natural tendency to think in terms of simplified mental models at an initial stage of the learning process, that help us to represent the original identities with the aim of facilitating and stimulating the task of understanding or resolution, and which later become implicit or tacit, continuing to influence, unconsciously, the interpretations and the solving decisions of the learner.

In general, learning is a complex process and there are different theories that attempt to answer the essential question: How do we learn in mathematics? Some of these theories claim that we create our conceptual systems using metaphors that can only emerge using language, indicating the social and supra-individual character of learning. Within this context it is also affirmed that metaphorical projection is a recursive mechanism through which culture perpetuates and reproduces itself in a system of concepts continuously growing [18]. Generally, it is recognized that the use of metaphors helps us to communicate effectively in our daily lives allowing us to understand complex topics, while conditioning our attitudes and our reasoning [19].

In this context it is natural for us to ask if there are connections between this type of metaphorical thinking and these unconscious, *tacit models* described by Fischbein [2] that could benefit the understanding of mathematical infinity in the university classroom.

2 Methods

In recent years, new theories have been developed [20], [21], [22], [23], [24], [25] suggesting there are alternative ways of formalizing mathematical ideas and highlighting the relevant role of different sensory modalities in the processes of abstraction, and in general, in the construction of concepts from simple to complex [26], [27]. These theories are supported by numerous studies that were carried out during the so-called "brain decade" (1990-2000) that bring out the functioning of the motor-sensory system of the brain in the development of conceptual knowledge and examine how brain physiological functions support cognitive processes [28], [29], [30], [31], [32].

In this work one of these interdisciplinary theories is used, which provide useful tools to deepen our understanding of these tacit, unconscious models related to the study of mathematical infinity. Based on the copious experimental evidence collected by neuroscience research and the embodiment cognition theory proposed by Rosch, Thompson and Varela [33], Lakoff and Núñez [20] introduced cognitive linguistics to examine the teaching and learning processes in mathematics, arguing that some of the constituent processes of cognition are based on and are derived from the interaction of the environment with the individual through his motor-sensory circuit, and as a result, abstract concepts such as mathematical ones can emerge through their perceptions and these bodily actions and interactions with the environment.

Thus, cognitive linguistics offers a set of techniques to study implicit conceptual structures in our experiences which, to a large extent, are unconscious, playing a fundamental role in the construction of mathematical ideas [20], [34]. From this perspective, images schemas [35] could evolve and lead to cognitive mechanisms known as *conceptual metaphors*, which constitute more or less conscious associations or links between different conceptual domains, in which the structure of the source domain infers a structure in the target domain. The origin of a *conceptual metaphor* can be an embodied experience of the physical world, as well as a previous existing conceptualization, creating a link that connects concepts or subdomains, outside or within mathematics, as shown in Figure 1.

Note that *conceptual metaphor*'s mappings do not involve random correspondences, but rather systematically preserve the image schema structure of the source domain in a way consistent with the inherent structure of the target domain, relating elements within each conceptual domain. By preserving inferential structure, *conceptual metaphors* enable a transfer of understanding that allows the comprehension of difficult, abstract, obscure or unconscious concepts in terms of those better known [36].

One of the main postulates of this theory is that *conceptual metaphors* are rather grounded in human embodiment and embodied experiences and are deeply entrenched in the way human think. Such metaphors are typically invoked without conscious recognition, being partially unconscious, becoming gradually conscious overtime [36]. From the perspective of cognitive neuroscience these metaphors are cognitive mechanisms arising through the simultaneous activation of distinct areas of the brain, each concerned with distinct aspects of experience that gives rise to neural conflation or the wiring together of neural circuits that were not connected in that specific way before.

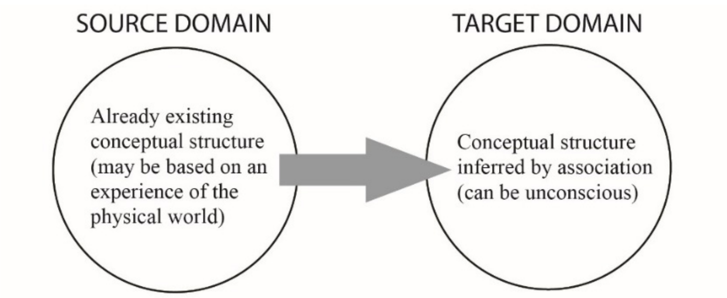

Fig. 1. The conceptual metaphor

3 Results and discussion

Some authors affirm [2], [34] that for students to overcome the obstacles posed by these *tacit models*, it is necessary to help them to become aware of their mental processes and to exert control over them. For example, Arrigo and D'Amore [9] recommend that to rectify the belief that in a longer segment there are more points than in a shorter segment (*tacit model* known as dependence), students be helped to separate from the unconscious model of the segment as a "necklace", whose "pearls" are closely arranged.

As we shall see *conceptual metaphors* plays a key role in explaining how mathematical infinity is grounded in our experience and therefore provides a mechanism to address these *tacit models* by showing us the way to overcome the obstacles posed by them.

We start our discussion noting that some of the difficulties faced by students in relation to this mathematical concept can also be studied, in part, through the difficulties that mathematicians of previous generations faced throughout history while trying to understand infinity and they are of a similar nature. Let us also highlight that, in general, three types of obstacles are distinguished in the learning of any mathematical concept: the ontogenic ones that originate in the characteristics of the student's development; the didactic ones that are a product of teaching; and the epistemological ones that are intrinsically related to the mathematical notion under study.

In this case, historical studies suggest some of the *tacit models* involved in conceiving mathematical infinity [8]. An example of this is the *tacit model* known as *inclusion*, provided by Galileo in the XVII century. While observing that natural numbers and perfect square numbers could be put into one-to-one correspondence, he could not reach any conclusion. For him, if the set of square numbers is a subset of natural numbers, then they should form a smaller collection than the set of natural numbers. Faced with this paradoxical situation, Galileo concluded that sets should not be compared when one or both had an infinite number of elements [9]. His reasoning was deeply rooted on the unconscious belief "the whole is greater than its part" which is a common notion of Euclid's Elements. Note that in this apparent paradox also appears the model called *slipping* by Duval [13].

It was by questioning precisely this unconscious model that Dedekind was able to give his definition of an infinite set in the XIX century. He stated that if the elements of two sets, finite or infinite, could be paired by a one-to-one correspondence, then they would have the same number of elements. But how was it that Dedekind was able to overcome the obstacle posed by this *tacit model*?

To answer this question, we start by noting that, in order to be able to characterize the notion of size (cardinal number) for infinite sets, it is essential to use the *conceptual metaphor* "same number as" is "pairability" [20] shown in Figure 2. This mechanism gives a precise metaphorical meaning to the comparison of the number of elements (that is, cardinality) of infinite sets. The everyday notions of "same number as" and "more than" are based on the experience we have with non-infinite, finite collections. When we consider that a set A (finite) has the "same number" of elements as a set B (finite), we mean that for each element of A, a corresponding element of B can be removed, and that no element of B remains "left-over".

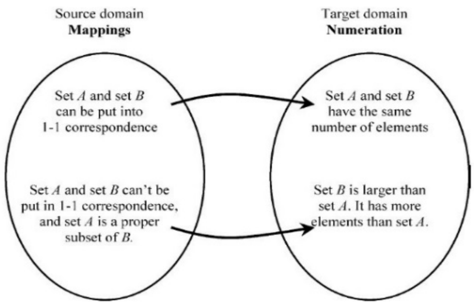

Fig. 2. The *conceptual metaphor* "same number as" is "pairability" [20]

If we extend the “left-over” idea to infinite sets and try to answer the question “Are there more natural numbers than odd numbers?” we can match the elements of both sets and conclude that there are more natural numbers than odd numbers, because in the set of natural numbers there are even numbers “left-over”.

However, these two sets are also “pairable” because we can put their elements in one-to-one correspondence. Therefore, “pairability” and “same number as” are two very different ideas that have the same extension for finite collections, but they are cognitively different, their inferential structures differ in important ways and do not have the same extension for infinite collections [37] as it is shown in Figure 3. Dedekind realized this and used the “pairability” instead of our everyday notion of “same number as”. Note then that in order to do that he had to actively and fully ignore the “left-over” clause that is unconsciously implicit in our ordinary notion of “more than” and by doing so, he was able to complete the metaphorical extension to conceive cardinality for infinite sets.

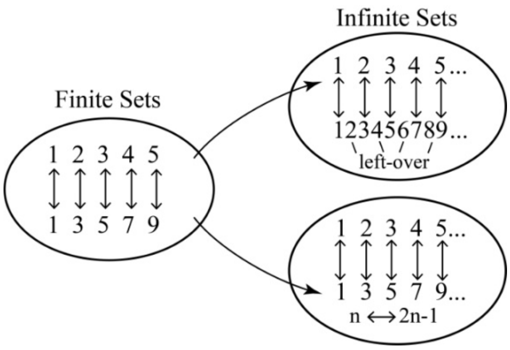

Fig. 3. Different extensions (“same number as” and “pairability”) for infinite collections

It is worth noting that this *conceptual metaphor* allows us to understand the processes of abstraction that are built metaphorically from contradictory conceptual structures by showing us, at the same time, the reasons for which the results of these processes seem counterintuitive to us. Thus, *tacit models* can be consistently used as the source domains of these *conceptual metaphors*, specifying obstacles and difficulties and pointing out the associations that must be considered to overcome these obstacles and to achieve an adequate understanding of these complex mathematical concepts.

Let us recall that Lakoff and Núñez [20] propose that the conception of actual infinity is based on the so-called basic mapping of infinity (BMI), through which, the processes that continue indefinitely are conceptualized in such a way that they achieve a final result, a kind of “metaphorical completion”. The BMI is a *conceptual blend*, a generalized form of *conceptual metaphor*, which also constitutes an association or link between different conceptual domains, as it is shown in Figure 4. Specifically, in this case, two source domains are connected to selectively “project” into a target domain where a new structure develops. The first source domain is a space that involves completed iterative processes. The second source domain involves endless iterative processes, and therefore characterizes processes related to potential infinity. In the integrated target domain, a structure emerges that is inferred to characterize the processes involved in actual infinity. As a result, the BMI is a human conceptual mechanism responsible for the creation of all kinds of actual infinities in mathematics, from infinite points in the plane to infinite sums, infinite sets and infinitesimal numbers and limits.

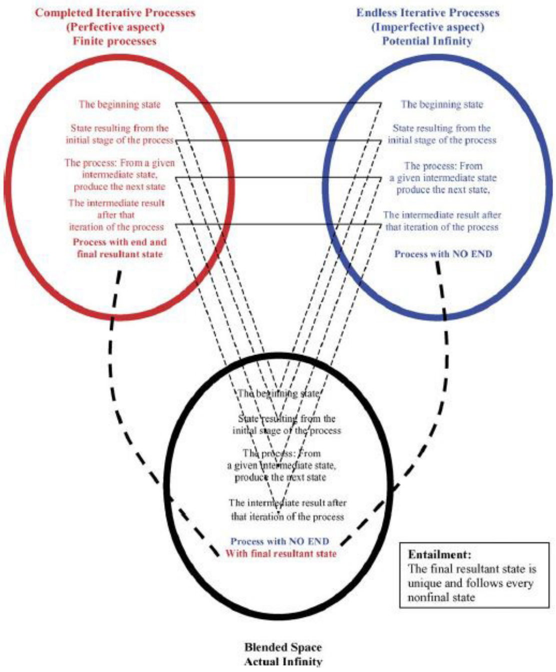

Fig. 4. The basic mapping of infinity (See Ref. [20])

Figure 4 shows the correspondences between the source domains and the projections towards the integrated space that constitutes the target domain. The correspondence between the two input spaces involves all elements except for the last one, the only element that fundamentally distinguishes a finite process from a potentially infinite one. This provides a conflict between a characterization of a process that explicitly has an end and a final resultant state, and a characterization that explicitly indicates a never-ending process with no resulting end state.

Thus, Galileo in the XVII century was faced with this conflict when he observed that the elements of the set of natural numbers and the set of perfect square numbers could be put into one-to-one correspondence, and he could not reach any conclusion that would have required completing the endless process [9].

Once again, the study of the BMI gives us insight into the conflicting structures that appear in this case, showing us that the *tacit model* known as *slipping* arise when no integrated space is formed from the two original source domains, which remain as they were, with their own local structures.

As Lakoff and Núñez [20] have pointed out, a crucial implication of the BMI is that the resulting final state is unique and follows every non-final state. Uniqueness comes from the source domain of completed processes where for any of these processes the resulting end state is unique. Let us also note that the resulting final state is final means that there is no previous final state. Similarly, there is no later end state of the process. That is, there is no other process state that results from the process completion and follows the process end state. For example, the *tacit model* known as *divergence* [38] which appears while studying the convergence of infinite sums, usually presented to students by Zeno's paradox on Achilles and the Tortoise, could also be explained in terms of these difficulties.

The *squeezing* of transfinite cardinals is another *tacit model* mentioned before which is present unconsciously in the thought processes of students when they affirm the set of natural numbers and the set of real numbers have the same cardinality. Cantor's ingenious diagonal argument to demonstrate the opposite of this, i.e., that there are more real numbers than natural numbers, also makes implicit use of the BMI [35]. Thus, once again the BMI provides a mechanism to address this *tacit model*.

As we have seen in our analysis, *tacit models* related to actual infinity could be understood through the study of the BMI. The transfinite cardinals, as well as Cantor's transfinite ordinals, and the infinitesimals in their actual sense, were "metaphorically completed" through this metaphor. Thus, obstacles posed by *tacit models* related to all these mathematical concepts could be overcome by the conscious construction of these associations, metaphorical mappings and projections. Conversely, it is shown that these models can be usefully and consistently understood in terms of the BMI, specifying conflicts, obstacles and difficulties that arise in each case.

Finally, we recall that from the point of view of the embodiment cognition theory, supported by the experimental evidence collected by neuroscience research, some of the constituent processes of cognition are based on and are derived from our motor-sensory perceptions, and as a result, the emergence of abstract concepts through cognitive mechanisms like *conceptual metaphors*, such as the ones characterizing mathematical infinity, are constrained by bodily-grounded experience. The appearance of these unconscious, *tacit models* is just a direct manifestation of these constraints. In the case of the *tacit models* related to actual infinity previously analyzed, these constraints are provided by container schemas for understanding finite collections and their hierarchies, by unconscious discriminating mechanisms genetically determined and kinesthetic experiences involved in size comparison and elements matching for finite collections [20].

In this context, using technology to engage students to help them overcome these limitations and difficulties would also play an important role. Learning in a digitally rich setting offers opportunities to exploit a wider range of perceptual-based experiences than traditional learning.

Technology could help us create blended learning environments [39] that stimulate thought processes leading to the construction of the appropriate metaphors in this case. For example, enhancing contextually designed activities may cause release from these embodiment constraints provided by container schemas and similar unconscious mechanisms underlying these *tacit models*. This release from embodiment may, in fact, be a seam around which learning may be mediated through encouraging different levels of engagement and reflection and new ways of thinking about mathematical infinity [40].

4 Conclusion

In the context of cognitive linguistic, conceptual metaphorical mappings and projections as *conceptual metaphors*, are non-arbitrary cognitive mechanisms through which the underlying obstacles and difficulties posed by *tacit models* that emerge as unconscious convictions in the study of mathematical infinity, could be addressed and overcame in a conscious way.

Furthermore, counterintuitive, unconscious ideas and paradoxes related to this mathematical concept are interesting from a cognitive perspective, because they help us to understand abstraction processes that develop from conflicting cognitive structures. Conceptual domains that are mapped metaphorically through these cognitive mechanisms can be consistently understood in terms of these *tacit models*. Thus, conversely, the conflicts underlying these unconscious *tacit models* are essential for the emergence of *conceptual metaphors* in relation to mathematical infinity and can be studied and stated precisely.

Moreover, this type of study that shows the connections between *conceptual metaphors* and unconscious, *tacit models*, benefit our understanding of mathematical infinity in the university classroom. It allows us to improve our teaching practice by designing activities aimed at achieving the development of *conceptual metaphors* relevant in each case, leading students to overcome these models. It also helps us to stimulate students to become aware, to reflect on the inconsistencies of their thoughts and intuitions regarding this mathematical concept. At the same time, it shows to students the validity of these inconsistencies, by revealing how our cognitive processes are constrained by bodily-grounded experiences determined by the morphology and the complexity of our human nervous system.

Using technology to engage students on these reflections would also help them master the learning processes related to mathematical infinity. The metaphor-based interactions, particularly the metaphorical mappings between physical and conceptual interactions in a digitally rich environment, could allow them to better understand this abstract concept. Nevertheless, there is a need for further research in the current context of digital technology in mathematics education to develop these approaches.

5 References

[1] Arrigo, G. & D'Amore, B. (1999). Lo veo, pero no lo creo: obstáculos epistemológicos y didácticos para la comprensión del infinito actual. Educación Matemática, 11(01), 5-24. http://funes.uniandes.edu.co/10174/

[2] Fischbein, E. (2001). Tacit models and infinity. Educational studies in Mathematics, 48, 309-329. https://doi.org/10.1023/A:1016088708705

[3] Tall, D. (1991). The Psychology of Advanced Mathematical Thinking. En Tall, D. (Ed.) Advanced Mathematical Thinking, Kluwer: Hollad, 3-21. https://doi.org/10.1007/0-306-47203-1_1
[4] Aristotle. (1985). The complete Works of Aristotle. The revised Oxford translation. In J. Barnes (Ed.). New Jersey, United States: Princeton University Press. https://press.princeton.edu/books/hardcover/9780691016511/complete-works-of-aristotle
[5] Bagni, G.T. (2004). Exhaustion argument and limit concept in the History of Mathematics: educational reflections. In Furinghetti, F, Kaiser, S. & Vretblad, A. (Eds.), Proceedings of HPM–2004, History and Pedagogy of Mathematics, Uppsala, 94–103. https://www.semanticscholar.org/paper/Exhaustion-argument-and-limit-concept-in-the-of-Bagni/bcf545e09e1e27c7bb41ba6f61a19c26e437eb15
[6] Moreno, L. & Waldegg, G. (1991). The conceptual evolution of actual mathematical infinity. Educational Studies in Mathematics, 22, 211-231. https://doi.org/10.1007/bf00368339
[7] Tsamir, P. & Tirosh, D. (1994). Comparing infinite sets: intuitions and representations. Actas del PME, 18 (4), 345-352. https://link.springer.com/article/10.1023/A:1003514208428
[8] Díaz-Chang, T. & Arredondo, E-H. (2021). Del infinito potencial al actual: un recorrido histórico a través de la metáfora conceptual. Revista Paradigma, 42(1), 106-132. https://dialnet.unirioja.es/buscar/documentos?querysDismax.DOCUMENTAL_TODO=Del+infinito+potencial+al+actual+un+recorrido+historico+a+traves+de+la+metafora+conceptual
[9] Arrigo, G. & D'Amore, B. (2004). Otros hallazgos sobre los obstáculos en la comprensión de algunos teoremas de Georg Cantor. Educación Matemática, 16 (2), 5-20. https://dialnet.unirioja.es/servlet/articulo?codigo=1159533
[10] Fischbein, E., Jehiam, R. & Cohen, D. (1994). The irrational numbers and the corresponding epistemological obstacles. Proceedings of the XVIII PME. 2. Lisboa, 352-359.
[11] Waldegg, G. (1993). La comparaison des ensembles infinis: un cas de resistance a l'instruction. Annales de Didactique et des Sciences cognitives, 5, 19-36.
[12] D'Amore, B. (2011). La didáctica del infinito matemático. En AA. VV., Memorias del XXIV Coloquio Distrital de Matemáticas y Estadística, Bogotá, Colombia. CD. ISBN 978-958-57050-0-5, 21-29. http://www.dm.unibo.it/rsddm/it/articoli/damore/746%20Coloquio%202011.pdf
[13] Duval, R. (1983). L'obstacle de dédoublement des objets mathématiques. Educational studies in Mathematics, 58, 335-359. https://www.jstor.org/stable/3482436
[14] Tall, D. (1981). Intuitions of infinity. Mathematics in School. https://www.semanticscholar.org/paper/Intuitions-of-Infinity.-Tall/2f002d7dfe2f6db4ebd2278fe33775f7747b7df9
[15] D'Amore, B. & Martini, B. (1997). Contrato didáctico, modelos mentales y modelos intuitivos en la resolución de problemas escolares típicos. Números, 32, 26-32. http://www.dm.unibo.it/rsddm/it/articoli/damore/746%20Coloquio%202011.pdf
[16] Dubinsky, E., Weller, K., Mc Donald, M. & Brown, A. (2005). Some historical issues and paradoxes regarding the concept of infinity: An APOS-based analysis. Educational studies in Mathematics, 58, 335-359. https://www.jstor.org/stable/25047191
[17] Bolzano, B. (1991). Las paradojas del infinito. D.F., México: Universidad Nacional Autónoma de México. https://www.sudoc.fr/087870428
[18] Sfard, A. (2008). On two metaphors for learning and the dangers of choosing just one. Educational Researcher, 27(2), 4-13. Cichocki, A. and Unbehaven, R., (1993). Neural Net-

works for Optimization and Signal Processing, 1st ed. Chichester, U.K.: Wiley. https://doi.org/10.3102/0013189X027002004
[19] Flusberg, P., Matlock, T. & Thibodeau, S. (2019). The role of metaphor in communication and thought. Language and Linguistics Compass, 13(5). https://doi.org/10.1111/lnc3.12327
[20] Lakoff, G. & Núñez, R. (2000). Where Mathematics comes from. Basic Books, New York, USA. https://doi.org/10.2307/3072449
[21] Tall, D. (2004). Building theories: The three worlds of mathematics. For the learning of Mathematics, 24, 1, 29-32.
[22] Edwards, L.D. (2011). Embodied cognitive science and mathematics. En B. Ubuz (Eds.), Proceedings of the 35th Conference of PME. 2. Ankara, Turkey, 297-304.
[23] Radford, L. (2013). Sensuous cognition. In D. Martinovic, V. Freiman & Z. Karadaq (Eds.), Visual Mathematics and cyberlearning, (pp. 141-162). New York, NY: Springer. https://doi.org/10.1007/978-94-007-2321-4_6
[24] Arzarello, F., Robutti, O. & Thomas, M. (2015). Grow point and gestures: looking inside mathematical meanings. Educational studies in Mathematics, 90 (1), 19-37. https://doi.org/10.1007/s10649-015-9611-5
[25] Wu, Z., & Jia, F. (2022). Construction and Application of a Major-Specific Knowledge Graph Based on Big Data in Education. International Journal of Emerging Technologies in Learning (iJET), 17(07), pp. 64–79. https://doi.org/10.3991/ijet.v17i07.30405
[26] Peechapol, C. (2021). Investigating the Effect of Virtual Laboratory Simulation in Chemistry on Learning Achievement, Self-Efficacy, and Learning Experience. International Journal of Emerging Technologies in Learning (iJET), 16(20), pp. 196–207. https://doi.org/10.3991/ijet.v16i20.23561
[27] Wang, J., & Zhan, Q. (2021). Visualization Analysis of Artificial Intelligence Technology in Higher Education Based on SSCI and SCI Journals from 2009 to 2019. International Journal of Emerging Technologies in Learning (iJET), 16(08), pp. 20–33. https://doi.org/10.3991/ijet.v16i08.18447
[28] Berntson, G. G. & Cacioppo, J. T. (Eds.). (2009). Handbook of neuroscience for the behavioral sciences, Vol. 1. Hoboken, NJ, US: John Wiley & Sons Inc. https://doi.org/10.1002/9780470478509
[29] De Smedt, B. (2014). Advances in the use of neuroscience methods in research on learning and instruction. Frontline Learning Research, 6, 7-14.
[30] Feldman, J. & Narayanan, S. (2004). Embodied meaning in a neural theory of language. Brain and Language, 89, 385–392. https://doi.org/10.1016/S0093-934X(03)00355-9
[31] Merkley, R., Shimi, A. & Scerif, G. (2016). Electrophysiological markers of newly acquired symbolic numerical representations: the role of magnitude and ordinal information. ZDM- The International Journal on Mathematics Education, 48, 3. https://doi.org/10.1007/s11858-015-0751-y
[32] Spüler, M., Walter, C., Rosenstiel, W., Gerjets, P., Moeller, K., & Klein, E. (2016). EEG based prediction of cognitive workload induced by arithmetic: a step towards online adaptation in numerical learning. ZDM- The International Journal on Mathematics Education, 48, 3. https://doi.org/10.1007/s11858-015-0754-8
[33] Rosch, E., Thompson, E. & Varela, F. (1991). The embodied mind: Cognitive science and human experience. MIT Press, Cambridge, MA. https://doi.org/10.7551/mitpress/6730.003.0005
[34] Gallese, V. & Lakoff, G. (2005). The brain's concepts: The role of the sensory-motor system in conceptual knowledge. Cognitive Neuropsychology, 22 (3), 455-479. https://doi.org/10.1080/02643290442000310

[35] Rohrer, T. (2006). Image Schemata in the Brain. En From Perception to Meaning: Image Schemas in Cognitive Linguistics, Beate Hampe and Joe Grady, eds., Berlin: Mouton de Gruyter. 165-196. https://doi.org/10.1515/9783110197532.2.165
[36] Johnson, M. & Lakoff, G. (2003). Metaphors we live by. The University of Chicago Press, Chicago. https://doi.org/10.7208/chicago/9780226470993.001.0001
[37] Núñez, R. (2005). Creating mathematical infinities: Metaphor, blending, and the beauty of transfinite cardinals. Journal of Pragmatics, 37, 1717-1741. https://doi.org/10.1016/j.pragma.2004.09.013
[38] Belmonte, J. L. & Sierra, M. (2011). Modelos intuitivos del infinito y patrones de evolución nivelar. Revista Latinoamericana de Investigación en Matemática Educativa, 14(2), 139-171. https://dialnet.unirioja.es/buscar/documentos?querysDismax.DOCUMENTAL_TODO=Modelos+intuitivos+del+infinito+y+patrones+de+evoluci%C3%B3n+nivelar
[39] Azizan, F. Z. (2010). Blended Learning in higher Education Institution in Malaysia. Proceedings of Regional Conference on Knowledge Integration in ICT. Recuperado de: http://ldms.oum.edu.my/oumlib/sites/default/files/file_attachments/odl-resources/4334/blended-learning.pdf
[40] Farr, W., Price, S., & Jewitt, C. (2012). An introduction to embodiment and digital technology research: Interdisciplinary themes and perspectives. NCRM Working Paper 2. https://eprints.ncrm.ac.uk/id/eprint/2257/4/NCRM_workingpaper_0212.pdf

6 Authors

Tamara Díaz-Chang, Ph.D.candidate, currently studying at the Graduate School, Department of Exact Sciences, Universidad de Los Lagos. Adjoint Professor at the Institute of Physical and Mathematical Sciences, Universidad Austral de Chile. Her research interests focus on applied mathematics, cognitive neuroscience and its applications to mathematics education (email: tamara.diaz@uach.cl).

Elizabeth-H Arredondo, Ph.D. Associate Professor at Department of Exact Sciences, Universidad de Los Lagos. Her research interests focus on the use of technology in mathematics education, problem resolution, geometric thinking, didactics of statistics and probability (email: elizabeth.hernandez@ulagos.cl).

Article submitted 2022-06-03. Resubmitted 2022-07-06. Final acceptance 2022-07-07. Final version published as submitted by the authors.