Design Considerations in Developing an Augmented Reality Learning Environment for Engaging Students in Covariational Reasoning

https://doi.org/10.3991/ijet.v18i11.38923

Otman Jaber[1,2(✉)], Osama Swidan[1], Michael N. Fried[1]
[1]Ben-Gurion University of the Negev, Program for Science and Technology Education, Beer-Sheva, Israel
[2]Al-Qasemi Academic College of Education, Baqa-Elgabiyah, Israel
jaberot@post.bgu.ac.il

Abstract—This study reports on findings of two design cycles of augmented reality environment intended to engage high school students in covariational reasoning. The study used a designed-based research method to develop and improve the learning environment. In this report, we present the initial design and discuss how it promoted students' engagement at elementary levels of covariation. Following this first cycle, we introduced a redesigned learning environment. We provide evidence of how the new design in the second cycle promoted students' engagement at advanced levels of covariational reasoning. Six groups of three 15- to 17-year-old students participated in the research. Using AR headsets, each group carried out two activities well-suited, in principle, to covariational reasoning. The students' interactions were video-recorded, and the theory of semiotic representation was used to analyze the degree of their engagement in covariational reasoning. The design emphasized multiple representations generally, and the compatibility between the explored phenomenon and its mathematical representations, specifically. Findings show that the design considerations in the second design cycle significantly improved the students' engagement at different levels of covariation, including advanced levels.

Keywords—covariational reasoning, representations, design principles

1 Introduction

Covariational reasoning has been shown to be important, if not crucial, for understanding mathematical concepts, function graphs, and functional relationships at all levels of schooling [1]. However, studies report difficulties in covariational reasoning among students from different grades and levels (e.g., [2–3]. Several studies emphasize the need for further research on developing students' covariational reasoning (e.g., [1–4]).

With the development of digital technologies in mathematics education, efforts have been made to design learning environments to advance students' covariational reasoning ([5–6]), including the use of simulations and the creation of new digital tools.

Studies in this direction indicate that utilizing *dynamic* tools fosters students' covariational reasoning (e.g., [7]). In addition, the results from these studies and others are consistent with theoretical findings that *multiple representations* are essential in mathematical learning environments and the ability to transform one set of representations into another is central for mathematical reasoning [8].

Among these dynamic tools, augmented reality (AR) technology has been investigated as a tool specifically for engaging students in covariational reasoning [9]. This technology provides a stage for modelling dynamic phenomena—phenomena in which covariation is inherent [10]—by providing real-time data of the phenomena, and juxtaposing the real phenomenon with a variety of mathematical representations (e.g., graphs, numbers). These affordances of AR, based on its distinctive ability to overlay dynamic phenomena with multiple mathematical representations, have indeed been shown to be beneficial for engaging students in covariational reasoning [11].

It is true that AR has been widely used in educational settings (e.g., [12–13]) during the last decade; however, most of the relevant research has been focused on using AR to visualize mathematical and scientific objects [14]. Other research centered on students' motivation when using AR [15] and spatial reasoning [16–17]. Much less research, though, has been dedicated to examining the use of AR for learning concepts connected to the mathematics of change [11]. In this study, we used AR not to visualize mathematical objects in 3D but to model dynamic real-world phenomena and juxtapose them with their mathematical representations. The study specifically aims to identify the design principles of the AR environment important for developing students' advanced covariational reasoning. It also aims to shed light on the role of juxtaposing real-world phenomena with their mathematical representation, which, as we have said, is one of the unique affordances of AR in learning mathematics of change.

Identifying the design principles of AR technology for learning mathematics of change concepts has theoretical, methodological, and practical implications. Theoretically, this study should shed light on the role of AR in learning mathematics through modeling real-world phenomena while simultaneously developing students' covariational reasoning. Methodologically, this study provides a useful example of employing Design-Based Research to design and study new digital technology to learn mathematics. Practically, this study offers design principles for educational software designers to develop and use AR in educational settings concerned with the mathematics of change.

2 Theoretical framework

Two theoretical perspectives form the backbone of this paper's approach to designing its AR learning environment. The first is the *covariational reasoning* framework which describes the intended mathematical content domain and the modes of reasoning connected to it. The second is the *theory of semiotic representation* which concerns the ways students learn and understand mathematical concepts in general as well as grounding the coordination and transformations of different representations. We used this framework to design the learning environment and examine the students' learning.

2.1 Covariational reasoning

Covariational reasoning is defined as the ability to hold in one's mind a sustained image of two quantities' values (magnitudes) changing simultaneously. Thompson and Carlson [1] outlined six levels of thinking with respect to covariational reasoning. (1) No-coordination; (2) Pre-coordination of values; (3) Gross coordination of values; (4) Coordination of values; (5) Chunky continuous covariation; (6) Smooth continuous covariation. Below we describe each level and exemplify it by referring to 'filling the bottle' discussed in [2]. The problem considers the relationship between the water amount that flows into the bottle and the height of the water.

At the first level, the person has no image of variables varying together; the person focuses on one or another variable's variation with no coordination of values, such as "the height of the water is rising in the bottle," or "more water is being added to the bottle." At the second level, the person can predict the change of each variable value separately but does not create pairs of values, such as "after a certain amount of water is poured into the bottle, the water level in the bottle rises." At the third level, the person perceives a loose link between the overall changes in the values of the two quantities, such as "as the height increases, the volume increases as well." At the fourth level, the person can match the values of one variable (x) to the values of another variable (y), thus creating a discrete set of pairs (x, y), such as "when the amount of water was 100ml, the water level was 12 cm." At the fifth level, the person may perceive that those changes in the two variables coincide and that they vary smoothly but only in separate domains. Such as, "the water level is rising for each increment of water added, including all values of volume and height between successive values, e.g., 100ml and 200ml." At the sixth level, the person can perceive an increase or decrease in the value of one variable as co-occurring with changes in another variable value in its entire domain and see both variables as a smooth and continuous change. Such as, "the volume and height of the water vary smoothly through intervals; simultaneously, within each interval the amount of water and height of water vary smoothly and continuously."

The first framework sets the theoretical principles for the mathematical concept intended to be learned. When digital technologies are used for designing learning environments, semiotic representations such as graphs, symbols, and numbers, which are essential affordances of digital technologies, should be considered.

2.2 Theory of semiotic representations

The theory of registers of semiotic representations, suggests a way to understand how mathematical knowledge is acquired through varying representations of mathematical objects and situations [18]. Representations and their interrelations or transformations play a central role in learning mathematics and understanding mathematical concepts. This importance stems first of all from the mere fact that mathematical objects are not directly accessible by immediate perception or instruments. The only way to access and address them is to use signs and semiotic representations [8–18]. Accordingly, this theory emphasizes that mathematical thinking is achieved through representations, as one has access to mathematical objects.

Semiotic representations consider systems of signs such as language, writing, numbers, graphs, and drawing. Such representations allow students to express their ideas and they are utilized as instruments to express ideas and communicate thoughts [18].

This theory considers representations subject to characteristics of semiotic representation systems, called registers. Registers must allow transformation operations: (1) treatment and (2) conversion. Treatment occurs in transformations between similar semiotic systems. For example: solving an equation. Conversion occurs in transformations between different semiotic systems. For instance, transforming from algebraic to graphic representation. Duval emphasizes that different mathematical representations belonging to different semiotic registers are crucial, and mathematical activity and understanding depend on the ability to move between such semiotic registers.

Duval points out four types of semiotic registers: discursive, non-discursive, mono-functional, and multi-functional. The discursive registers refer to languages (oral and written) that express meaning units of thoughts and thought operations. Hence, these are process-oriented. The non-discursive registers display visual objects. Most processes take the form of algorithms within a monofunctional semiotic system, while within a multifunctional semiotic system, the processes can never be converted into algorithms [8].

Figure 1 illustrates the four registers of semiotic representations.

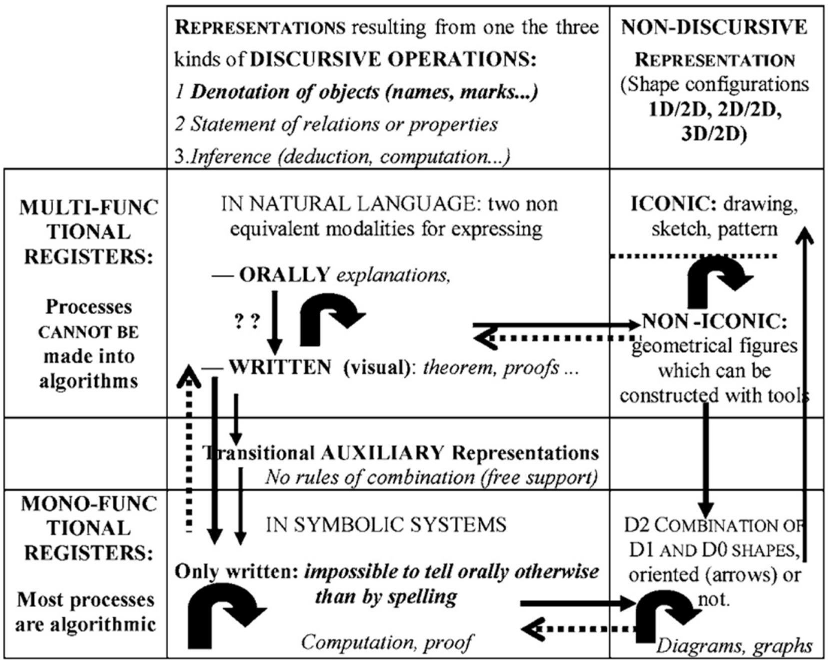

Fig. 1. Duval's four registers of semiotic representations—adopted from [8]

Covariational reasoning considers the ability to envision changes in variables (discrete or continuous) that vary simultaneously. But these variables and their covariational relations are not immediately evident in the real-world dynamic situations to which they apply. They must be made to appear to us by way of representations, particularly visual representations. AR technology comprises several types of representations, such as graphs and numerical measurements, that may include modelling dynamic situations. Such representations allow students to utilize them as instruments to communicate thoughts and engagement in covariational reasoning.

In addition, Duval's framework allows us to examine the evolution of students' covariational reasoning through the lens of registers by addressing the way students coordinate between registers (treatment, conversion). AR is an ideal context for observing these things because it naturally combines different registers. In our study, we consider three types of representations: (1) graphical representation, which refers to three kinds of graphs; points, segments, and continuous, (2) numerical representation, which includes a table of values, and (3) realistic phenomena, that refer to experiments with real-world phenomena in which a covariation concept is inherent. It may be objected that the latter is not a true representation in the way that a graph or table of values is. But one must realize that the real-world phenomena is still being seen within the frame of the AR glasses and its objects are marked and situated: it is presenting a circumscribed view of physical reality. For this reason, we include it as a representation, even though it is not of the type typically referred to in Duval's work.

3 Research questions

1. What are the design principles that engage students at advanced levels of covariational reasoning?
2. How do students interact with semiotic registers to achieve advanced levels of covariational reasoning?

4 Methodology

4.1 Research context

This study is a part of large project, focusing on technological design that considers real-life phenomena and their mathematical representations. The paper at hand emphasizes design principles to promote students' engagement in covariational reasoning. The design addresses three layers: technology, tasks, and AR tools and objects within the learning environment.

Research-based resources for designing AR learning environment aimed specific mathematical modes of thinking such as covariational reasoning are fairly sparse. In order to supply some basic design principles, we adopted an approach along the lines of design-based research (DBR) [19]. Although DBR as it is usually executed takes into account at least three cycles of design, in this study, we present only two cycles. We show that *even in the second cycle* of the design, a significant improvement of the students mathematical reasoning can already be discerned. Further, the design considerations planned for the third cycle, which grew out of the second cycle, mainly related to non-technological aspects (as we will elaborate in the last section). In general, the method we adopt here entails performing design cycles of technology, tasks, laboratory experiments, analysis, and reflection: analysis and reflection on the process provide feedback for *each* cycle. Analyzing each cycle allows us to shape pedagogical and technological design principles that promote learning and engagement in covariational reasoning. Such actions are usually addressed through three phases: (1) preparation and design, (2) teaching experiments, and (3) retrospective analysis [19]. These phases are explained in sections (4.2, 4.3, 4.4). The method we adopted in this paper is appropriate for AR technology not developed specifically for the learning of covariational reasoning: our design task involves the transformation of AR technology into a tool for this purpose, adapting and readapting it in light of the particular demands of covariational reasoning.

As the semiotic representations approach guides the design of AR environment, we assume that AR allows for the design of several representations. We also realize that the design alternations characterizing design-based research are addressed through changing representations according to indications of engagement in covariational reasoning. In addition, the task and activity design with AR, namely the educational goal, addresses the covariation concept.

4.2 Preparation and design

The initial design of the AR learning environment. This is a first step phase that is geared towards developing a theory for learning the covariation concept in an AR environment. Hence, engaging students in covariational reasoning was determined as a learning goal. Representations, provided by AR, as well as real world phenomena, have been addressed as means to support such engagement. In addition, inquiry tasks have also been developed as a pedagogy to support the learning process. Hence, the learning environment design considers the AR technology, the tasks, and real-world phenomena.

(i) <u>AR technology</u>: the AR design includes representations of graphs (points, segments) and tables of values with numerical measurements. AR models the real-world phenomenon and represents it as mathematical representations juxtaposing the dynamic objects of the phenomenon (e.g., see Figures: 2b, 3b, 4), which suggests opportunities for engagement in covariational reasoning.

(ii) <u>Tasks</u>: the task design considers a set of two-phase inquiry tasks; conjecturing and experimenting. The tasks are motivated by the AR technology design and guided by two experiments inspired by the real-world phenomena: (1) a Hooke's law activity (Figure 2a) examines the relationship between mass and the elongation of a spring, (2) a Galileo experiment (Figure 3a) examines the time-distance relationship as a cube slides down an inclined plane.

(iii) <u>Real-world phenomena</u>: This considers continuous (Galileo) and discrete (Hooke's-law) phenomena in which the covariation concept is inherent.

The environment also includes physical tools and objects essential for experimenting (e.g., weights, springs, inclined plane).

Example of Hooke's law activity (first cycle):

<u>*Small cubes*</u>

Hypothesis: What do you think will happen if you add 1, 2, 3, … small cubes? Discuss your conjecture/hypothesis and write it down.

Experiment: Conduct an experiment with your AR-device to check your conjecture.

Changing the cubes
You see different cubes now.
What do you think will happen if you change the cubes?
(What changes? What remains the same? No combination of different cubes.)

Big cubes
Hypothesis: What do you think will happen if you add 1, 2, 3, … big cubes? Discuss your conjecture/hypothesis and write it down.
Experiment: Conduct an experiment with your AR-device to check your conjecture.

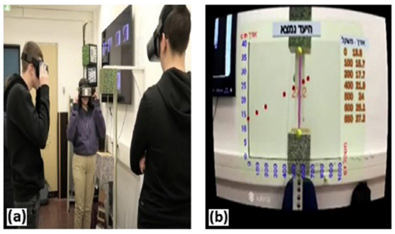

Fig. 2. (a) Hooke's law activity. (b) phenomenon as seen from AR headset

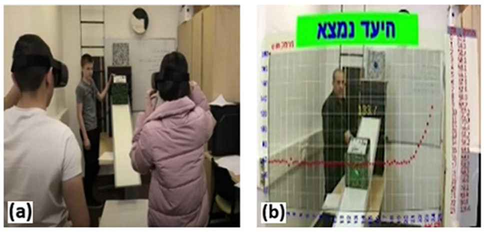

Fig. 3. (a) Galileo experiment. (b) phenomenon as seen from AR headset

Design principles of the first cycle. We addressed general theoretical design principles for the first cycle as a first step. Then, we developed more specific design principles for the next cycle drawing on that process. In this paper we focused on the technological design.

Technology design principle: the technology design includes several representations. It comprises two graphical representations (points, segments), and numerical representations (measurements) juxtaposing the physical phenomena, and represented in a table of values (see Figure 4).

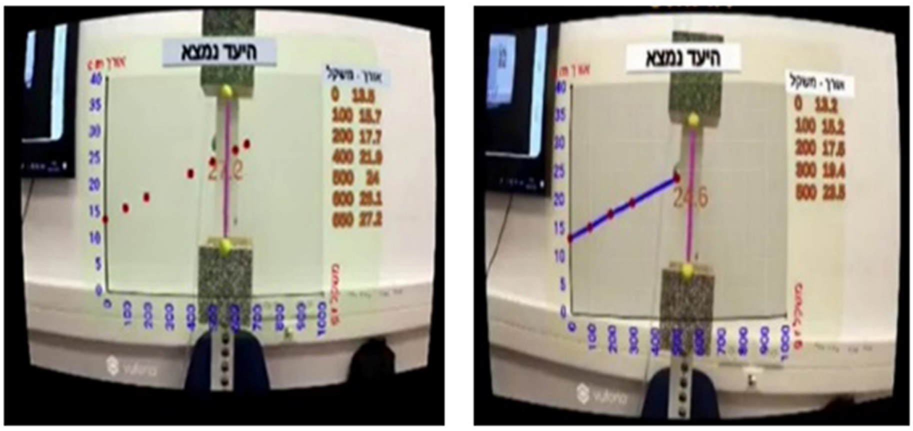

Fig. 4. Numerical representations and graphs (points, segments) juxtaposing the physical phenomenon as seen through AR headset in Hooke's-law activity

4.3 Teaching experiments

This phase considers an intervention process by experimenting with the developed theory in a learning environment. This phase suggests opportunities to examine to what extent the previous design promotes students' engagement in covariational reasoning.

Participants and process: Six groups of three 15- to 17-year-old students participated in the research. They had studied linear functions in the 8th grade and quadratic functions in the 9th grade. The meetings were held at the Ben-Gurion University of the Negev. Each session lasted about 180 minutes.

Each group conducted a set of experiments addressing two main tasks on dynamic situations that consider the covariation concept (Galileo, Hooke's law). They followed inquiry task-sheets in which they first raised conjectures on relationships between two variables of a real-world phenomenon (e.g., time-distance in Galileo experiment), then they checked their conjectures by experimenting, using the AR headsets. The mathematical representations juxtaposing the dynamic phenomena and provided by AR create opportunities for engagement in covariational reasoning. The students received explanations about the tasks and the use of AR technology.

Data collection: Data was collected through video recordings documenting all actions and interactions in the learning environment. All of the students' written notes (drawings, insights, etc.) were collected. The researcher's role was limited to monitoring and controlling the activity process, and illustrating unclear points. The AR virtual data of each student were reflected on a large screen. This allowed the researchers to follow what each student observed and better understand their explanations and arguments concerning what they observe.

The teaching experiments were integrated within two of the three cycles that guided this paper. With several experiments supported by multi-phase inquiry tasks for six groups of students, both cycles provided solid data for reaching insights on design principles that promote students' engagement in covariational reasoning.

4.4 Retrospective analyses

This phase considers the data analysis process and refinement of the hypothesized learning-environment design. The results of such a retrospective analysis mostly feed a new design phase (Bakker & van Erde, 2015), that eventually helps us shape technological design principles that promote engagement in covariational reasoning.

First, we describe general analysis steps, then we exemplify the analysis process using the theoretical frameworks.

The following steps illustrate the data analysis process:

1. Transcripts of the video recording of the observations were prepared.
2. Thompson and Carlson's [1] framework and the inductive-deductive approach [20] were used to identify covariational reasoning statements and determine categories and their levels.
3. Data on students' covariational reasoning statements were collected. Afterwards, we used Table 1, below, to sort the statements according to the components: statement level, the activity type, and the task number during which the statements were made.

This allowed us to follow the frequency, level and origin of each covariational reasoning statement.

Table 1. Statement' categorization table

	Tasks	Covariation Levels					
		L1	L2	L3	L4	L5	L6
Hooke's law experiment	T1						
	T2						
	T3						
	…						
Galileo experiment	T1						
	T2						
	T3						
	…						

4. We followed the distribution and level of covariational reasoning statements. Then we analyzed the situations or possible factors that may have promoted\hindered students' engagement in covariational reasoning, such as the technology, the task, the experiment, the type of phenomenon (continuous, discrete). Then, we examined how these factors related to the semiotic representation system. Similarly, we monitored the design principles and examined whether they supported or hindered engagement in covariational reasoning. For Instance, in the technology design, we followed the virtual representations, such as graphs of points and segments, and a table of values, and examined how they contributed to the students' engagement in covariational reasoning. For example, whether the students turn to this form of representation when they engaged in covariational reasoning, or whether students' covariational reasoning evolve through transformations between representations?
5. We examined factors and representations that promoted (or hindered) students' engagement in covariational reasoning, and considered them to determine design principles for the next cycle. For example, factors and representations that promoted engagement in covariational reasoning were considered as design principles. In contrast, factors, or representations, that hindered engagement in covariational reasoning were examined, then modified or eliminated.
6. We determined the design principles for the next cycle.
7. To provide more insight into the students' engagement in covariational reasoning and their interactions with semiotic registers, we also used statistical description for comparing findings in both cycles.

The following example illustrates how we used the Thompson and Carlson framework [1] to categorize a statement on covariation levels. In addition, we illustrate how we used Duval's framework considering registers and transformation between them.

The excerpt refers to a Hooke's law activity, in which the students used the AR headset to explore the mass-length relationship.

Excerpt 1:

549	Sagi	You see in the graph more as if ... such tangible ... you see more how the slope increases. you see that from one point to another as time increases from point to point, the distance increases… and we also observe that in the table
550	Noam	I also think so.... in the table we really see in points…the time and distance... and in the graph, we really see the change in distance at any point of time... as if you really see the whole relationship ... and the whole gradation.
551	Sagi	Yes, what we said is that from point to point the slope just gets steeper… because the acceleration increases.

Table 2 illustrates how we used the theoretical frameworks of both Thompson [1] and Duval [8] to analyze covariational reasoning statements and transitions between semiotic registers for the previous excerpt.

Table 2. Analysis model applying Thompson and Duval's frameworks

Covariation		**Representations**	
Line\ Variables	**Level\Description**	**Transition Description**	**Transformation**
[549] time-distance	L3: describing general relations and loose connections as overall changes between time and distance.	Interacting with the graph, then with the measurements in the table of values	Conversion: graphic-symbolic
[550] time-distance	L6: describing continuous distance changes in each point of time domain	Interacting with the graph, then with the measurements in the table of values	Conversion: Symbolic-graphic
[551] slope-acceleration	L3: describing general relations and loose connections between the graph's slope and acceleration.	Interacting with the graph's slope, then with the real-world	Conversion: Graphic-real-world

5 Findings

This section addresses both research questions on the students' engagement in covariational reasoning and their interactions with semiotic registers. First, we present insights and findings that arose following reflecting on the first cycle. Subsequently we address the redesigning of the learning environment and the findings of the second cycle.

5.1 Cycle 1

Reflection on the cycle. In the first cycle, we identified opportunities and difficulties to be taken into consideration during the second design cycle.

Opportunities

Following cycle 1, the students were engaged in elementary levels of covariational reasoning. From the technological aspect, the students turned to the available representations (graphs, table of values, numbers) during their engagement in covariational reasoning. They addressed the numerical data in the table of values and the graphs to explore the relationship with the observed phenomenon. Moreover, the juxtaposition of the real-time data and virtual representations with the real dynamic objects allowed the students to engage in covariational reasoning by covarying physical and virtual objects. Both activities, Hooke's law and Galileo, inspired students' engagement in covariational reasoning.

Following these insights, we preserved the following aspects within the design principles that mainly consider the aspect of different representations:

(1) Preserve both activities (Hooke's law & Galileo), (2) Preserve the virtual representations: graphs (points, segments), table of values, numerical measurements.

Difficulties

Despite the general contribution of the AR environment in engaging students in covariational reasoning, they engaged only at elementary levels of covariational reasoning. The students encountered difficulties that in fact relate to the same technological domain.

The technological problems included, first of all, *object identification problems.* These arose from the system whereby the AR technology looks for codes marked on an object to determine its position, extension, distance from a reference point: the ability of the AR device to identify the code is sensitive to environmental factors such as the distance or angle between the device and the object, lighting, and so on. Another problem is *data mirroring*, where the data provided by AR and observed by the students through their AR headset was not available to the researchers. Hence, understanding the entire scene was limited. *Data display*, a third problem, concerns improper data display (small size, data location, graph disturbance). The students were not familiar with *using AR*. Hence, more practice with AR is required. Bug problems and continuous graph design challenges have not yet been solved. This type of graph representation is essential and compatible for modeling continuous phenomena.

The following Table 3 summarizes the main technological design difficulties that arose in cycle 1.

Table 3. Technological design difficulties in cycle 1

	Technological Design Problems					
Problem	Identification	Data mirroring	Data display	Use of AR	Continuous graph	Bugs
Frequency	Frequent	Not available	Non-frequent	Frequent	Not available	Frequent

The difficulties that arose in cycle one mostly relate to representational aspects that entail either new designs (continuous graph, reflection phase) or modifications that are essential to enhance the representation system to provide a better experience of the learning process.

Students' covariational reasoning: findings from cycle 1. Results show that the first design cycle engaged the students at elementary levels of covariational reasoning.

Table 4 presents the frequency of covariation statements, categorized into levels, as they emerged in the course of the two activities.

Table 4. Frequency of covariational reasoning levels in Hooke's law and Galileo experiments in cycle1

	Low Levels		Medium Levels		Advanced Levels	
Covariation Level	**L1**	**L2**	**L3**	**L4**	**L5**	**L6**
Hooke's law activity	3	4	20	3	1	0
Galileo experiment	0	8	13	0	0	0
Total	3	12	33	3	1	0

According to Table 4, about 98% of covariational reasoning levels range between low levels (28.9%) and medium levels (69.2%). Advanced-level covariation was rare (1.9%). Students engaged more in covariational reasoning with the Hooke's law activity (59.6%) compared to the Galileo experiment (40.4%).

The results in Table 4 show that the students' engagement at medium levels of covariation was prominent in level 3. In contrast, engagement at level 4 was limited in the Hooke's law activity and nonexistent in the Galileo experiment. In the Hooke's law activity, students coordinated mass with the spring's length and created a few pairs of mass-length values. However, in the Galileo experiment, students were not engaged in coordinating the values of variables.

5.2 Cycle 2

Redesign of the AR learning environment. Following the reflection on the first cycle, we first addressed the technological problems. The technology with the representations was significantly improved. That was a crucial step, in line with the role of representations in shaping thinking. Therefore, to promote engagement in covariational reasoning, including at advanced levels, we emphasized the multiple representations and the compatibility between the explored phenomenon and its representations, which supports conceptual understanding [21]. For example, discrete phenomena, such as Hooke's law, should be explored with different representations, particularly with a compatible graph representation, such as points. This provides better opportunities for students to explore the phenomena with appropriate representations and compare between representations. Hence, in the Galileo experiment, we designed a continuous graph that may be utilized for continuous phenomena. The simultaneous change in continuous phenomenon variables and the smooth continuous graph formation may foster the emergence of advanced covariational reasoning levels. The learner may envision the changes in both variables as occurring smoothly and continuously according to Thompson and Carlson [1]. The continuous graph and other representations may draw students' attention to the synchronous changes in representations, compare between them, and eventually be better engaged in covariational reasoning.

The appropriate tasks were also considered to motivate the learner to explore and connect the different representations. For example, they direct students to explore the same phenomenon with different graph representations and draw students' attention to exploring and predicting the graph's changes and continuation.

Engagement in covariational reasoning: findings from cycle 2. Results show that the second design cycle engaged the students at different covariation levels, including advanced levels. Most levels of covariational reasoning focus on the gross coordination of values level, where students perceive a loose link between the overall changes in the values of two quantities.

Compared with the first cycle, findings in the second cycle show a more noticeable engagement of students at the coordination of values level. Indeed, they indicate a significant growth in the students' engagement at the chunky and smooth continuous covariation levels, as advanced levels of covariation.

Table 5 below shows the frequency of covariation statements, categorized into levels, as they emerged during the learning process in the Hooke's law activity and the Galileo experiment.

Table 5. Frequency of covariational reasoning levels in Hooke's-law and Galileo experiments in cycle 2

	Low Levels		Medium Levels		Advanced Levels	
Covariation Level	**L1**	**L2**	**L3**	**L4**	**L5**	**L6**
Hooke's law activity	2	16	80	31	7	7
Galileo experiment	1	36	35	2	12	11
Total	3	52	115	33	19	18

According to Table 5, about 84.6% of the covariational reasoning levels range between low levels (22.95%) and medium levels (61.65%), while (15.4%) were advanced levels. In addition, students engaged more in covariational reasoning during the Hooke's law activity (59.6%) than the during the Galileo experiment (40.4%).

Results show that the students' engagement at level 1, as a low level of covariation, was rare, while engagement at level 2 was noticeable. In addition, engagement at medium levels of covariation was prominent in level 3. In contrast, engagement at level 4 was limited in the Galileo experiment but significant in the Hooke's law activity, where students coordinated mass with the spring's length and created pairs of mass-length values. However, in the Galileo experiment, the students' engagement in coordinating variables' values was rare.

Identification of conversion transformations. The analysis of the students' engagement in covariational reasoning indicated the emergence of several transformations of conversions, made in registers of different semiotic representations. Such transformations have been identified at different levels of covariational reasoning. The following excerpts illustrate how conversions related to the evolution of advanced levels of covariational reasoning.

The following excerpt refers to the Galileo experiment as students discuss their insights on the relationship between the plane's inclination and the time-distance graph. It illustrates how they shift between covariational reasoning levels and the mathematical representations.

Excerpt 2

1635	Noam	When we changed the inclination, there was an acceleration
1636	Sagi	Acceleration is there all time; the question is, is it constant or varied?
1637	Noam	It is constant everywhere. It is a constant acceleration motion since we add 20cm for each second. I think it (acceleration) was still stable. Simply, when the inclination was down, instead of rising 20cm per half a second, it rose 10cm in each second, so that if here (drawing=graph) it was 30cm, 40cm. then at 4(seconds) it will reach here, and at five it will reach here, then they (points) will connect like this… (stressing line). It is still not a straight line.

In [1635], Noam covaries the inclination of plane and acceleration. She envisions both variables' values varying asynchronously—one variable changes, then the second variable changes. Hence, she demonstrates a pre-coordination of values level of covariational reasoning. In [1635–1636], Noam and Sagi interact and move between real-world representations, since the plane's inclination and acceleration are variables embedded in the Galileo experiment. We assume that as a treatment transformation.

In [1637], Noam converts to numerical representation to justify the constant acceleration "we add 20 cm for each second." She addresses proportional (or linear) relations between time and distance. For linear relationships between two variables, differences in the values of one variable are always in the same proportion as the difference in the values of the other. Noam's proportional relation indicates coincide changes among separate time-distance domain values, namely engagement in the chunky continuous covariational reasoning. Then, Noam continues her argument that acceleration is still constant even when the inclination is low, she sticks to numerical representation as she suggests another proportion "it rose 10cm in each second." Subsequently, Noam applies such proportional relations and turns to graphic representation, where she creates several time-distance points and connects them with a continuous line (see Figure 5). This indicates that Noam seems to be able to covary between any point of time, showing engagement at the sixth level of covariational reasoning. Ultimately, Noam relies on the graph, obtained by proportions, to justify that it is non-linear.

We notice that the treatment transformation within the real-world register has emerged as Noam was engaged at the second level of covariational reasoning, when she discussed her insights with Sagi [1635–1636]. In addition, engagement at advanced covariation levels was accompanied by conversion transformation among real-world [1635], numeric and graphic registers [1637].

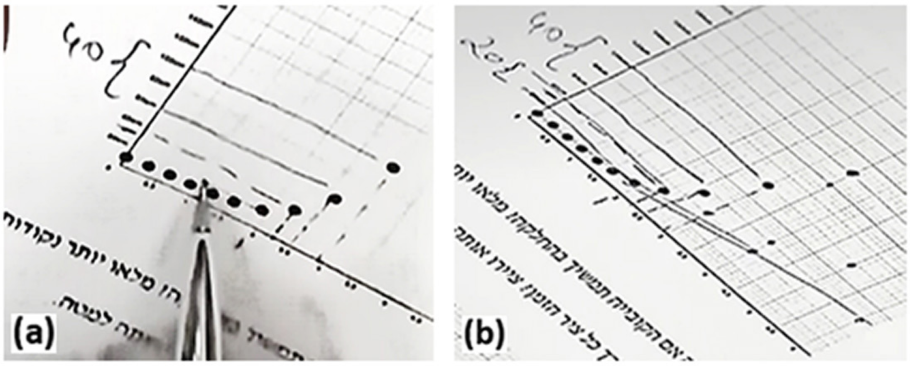

Fig. 5. Time-distance graph with: (a) high inclination (b) low inclination, using proportion

Excerpt 3

The following excerpt refers to the Hooke's the law activity, as students used the AR headset. It illustrates how students shift between semiotic registers as they are engaged in different levels of covariation, including advanced level.

320	Dennis	The spring has elongated
321	Shaked	I see 5.4…mm...it is writing 5.4
322	Dennis	Each time we add mass it elongates …How much you have at 100 (mass)?
323	Nir	I have 5.5…you Dennis 5.6 and you Shaked 5.4?
324	Shaked	At 100gr I got 5.4cm… Nir, has the red point on the graph changed? I mean, did it remain on the same line?
325	Nir	Yes, it remains on the same line, but it gets longer…. for each 100 the length of the spring increases by 0.3

In [320–321], Dennis and Shaked describe variations on the spring (level 1). As Dennis interacts with the spring "the spring has elongated", it seems that he interacts with the real-world phenomenon, while Shaked interacts with the numeric representation in the table of values "I see 5.4." Then Dennis, in [322] describes a general relation between mass and the spring's length (level 3) "each time we add mass it elongates." Nir, in [323] also interacts with the table of values and tells his mates the spring's length at 100 gr mass "I have 5.5." He also refers to the points that his classmates got earlier. Here, the students are engaged at the level of coordination of values, as they create pairs of mass length points. In [324] Shaked turns to the graph representation and asks Nir about the (mass, length) point that emerged on the same line (graph) "did it remain on the same line?" Nir who interacts with the graph and table of values in [325] refers to proportion relation as he is engaged at the fifth level of covariation and suggests a law of correspondence between mass and length "for each 100 the length of the spring increases by 0.3."

The excerpt shows that as the students were engaged at low and medium levels (L1, L3, L4) they mainly interacted with the real world and numeric registers. Lines [321–323] suggest treatment transformation as students mainly move between numeric registers. However, lines [324–325] suggest a shift to graphic register, which indicates a conversion transformation between a numeric and graphic registers.

Following the students' engagement in covariational reasoning and their interaction with semiotic registers, it seems that engagement in advanced covariational reasoning implies several conversion transformations, where graphic, symbolic (algebraic, numeric), linguistic and real-phenomenon registers are noticeable, see Excerpts 1, 2 and 3.

6 Discussion

The aim of this study was to identify the design principles of the AR environment towards fostering students' covariational reasoning and simultaneously, gaining some insight into how students engage with semiotic registers in connection with covariational reasoning. The initial design of the learning environment was motivated by connecting real-world phenomena that lend themselves to covariational reasoning with their mathematical representation. The hope was, moreover, that the design process together with the *theory of semiotic representation* which concerns the ways in which students learn and understand mathematical concepts in general would suggest how a well-designed AR learning environment might enrich students' levels of covariational reasoning in line with the scheme proposed by Thompson and Carlson [1]. The research focused on two design cycles.

Findings showed that the first design cycle engaged students at elementary levels of covariational reasoning, while the second design cycle showed an overall significant development of students' engagement at all covariational reasoning levels, including advanced levels, that were almost nonexistent in cycle 1. In this discussion we will address the role of AR features and the multiple representations in said improvement in light of the design cycles described in the paper.

6.1 AR features and representations

Analyses gave some indication that AR technology seemed to afforded the students' engagement at different levels of covariational reasoning, basically at the medium and advanced levels. On one hand, all low-level covariations emerged in the conjecturing phase in the task, before using AR in their learning. On the other hand, medium and advanced-level covariations were significantly identified during, or after, using the AR headsets. This shows the potential, at least, of AR technology to promote medium and advanced levels of covariation.

Monitoring the students' engagement in covariational reasoning showed that they interacted with the virtual and physical presentations demonstrated through AR. They referred to the table of values and graphs describing relations between variables, and they were able to connect the real-world phenomena they could see and the mathematical representations of the phenomena. AR affordances described here and reported elsewhere appear to support such actions: there are studies, for example, that point to the role of virtual objects in attracting learners' attention and motivating their interactions [22–23]. Bujak et al. [24] have also indicated that aligning information in time and space can help learners connect otherwise disconnected pieces of information; this too may account for the way students were observed to connect the dynamic phenomenon seen through the headset and the aligned mathematical representations, ultimately engaging them in covariational reasoning.

6.2 Design affordances

In addition to AR features, the design of representations also contributed to students' engagement in covariational reasoning. The findings show that engagement at different levels was noticeable in both activities, the Hooke's law and the Galileo experiment. That may be attributed, basically, to the design of different and appropriate representations.

Multiple representations: The multiple representations (table, numbers, graphs) contributed to the students' engagement in all covariational reasoning levels. The students frequently turned to the table of values as they engaged at low and medium levels. The two-column table seemed to be helpful for following and describing variable's variations, inferring general relations, or creating pairs of variable's values, which may be characterized as medium or low reasoning of covariation. At advanced levels, the students mainly focused on the graphs. For example, with the chunky graphs, the students described the variations between the boundaries of the chunks and the appropriate domains inspired by the table of values, which eventually engaged them in chunky continuous covariation. Similarly, with the continuous graph, students described the simultaneous variables' variations in each point over the whole variables' domain (see Excerpt 1,[550]).

The contribution of representations in engaging students in covariational reasoning, is supported by their essential role for meaning making and deep reasoning as reported in literature (e.g., [8–25]), which underpins their role within the design principles.

Compatibility between phenomenon and representation: Following the students' engagement in covariational reasoning, it seems that the compatibility between particular characteristics of the real-world phenomenon and its mathematical representation may also affect covariational reasoning levels. We found that engagement in the coordination of values level was noticeable with discrete phenomena (Hooke's law) when represented with the graph of points. However, with a continuous phenomenon, such as the Galileo experiment, the students were not engaged at the same level, although the phenomenon was modeled with different types of graphs. It seems that the short time experiment and the rapid appearance of the dynamic graph attracted their attention rather than the fixed table for creating pairs of values, and eventually engaging at the coordination of values level. We also found that continuous phenomena represented with a continuous graph contributed to the students' engagement at advanced levels of covariational reasoning. About 35% of advanced-level covariation statements (39) referred to continuous phenomena that were represented with a continuous graph, while 65% belonged to other representations and phenomena. This result supports the design of a continuous graph that is integrated with continuous phenomena to promote covariational reasoning. This finding is in tune with Cai [26], who argues that using appropriate pedagogical representations is important for explaining concepts, relationships and connections. Lee and Lee [21] also emphasize the critical role of appropriate representations for successful mathematical learning. They argue that representations should also be connected and consistent for supporting conceptual understanding.

6.3 Interaction with semiotic registers

This section refers to the second research question: How do students interact with semiotic registers to achieve advanced levels of covariational reasoning?

Analysis of the semiotic registers that emerged during the students' engagement at advanced levels of covariational reasoning indicates that students referred to different registers inspired by the designed representations, such as graphs, table of numerical values. Conversion transformations were identified among graphic, symbolic (algebraic, numeric), linguistic and real-phenomenon registers. Such conversions emerged as the students were engaged in advanced covariational reasoning.

Focusing on the students' interactions with the semiotic registers indicated two interaction situations: (1) during experimenting while using AR and (2) after experimenting without using AR.

Interaction while using AR: The interaction was characterized as visual. Engagement at advanced levels of covariational reasoning was mainly identified through conversion transformations, as students shifted between either numeric and graphic registers, or real-world and graphic registers.

Most of the advanced-level statements referred to the following situations: (a) applying rate of change (or ratio-proportion) relations, where the students transform between real world phenomenon and graphic, or numeric registers (e.g., Excerpt 2 [1637], (b) describing changes on a graph within chunks of numeric domains, where students transform between graphic and numeric registers, (c) describing a continuous change of graph on each point, or along the entire domain, where the students shift between numeric and graphic registers, providing further instances of conversion transformations, (e.g., Excerpt 1[550]).

Interaction after using AR: Here, the interaction was characterized as interpersonal and thus demonstrated a social aspect of AR and semiotic registers in the development of covariational reasoning. The emerging registers in this situation were inspired by the representations addressed *together* during experimenting and while using the AR headset. The students created drawings, shared and discussed data that were observed through the AR. They worked with graphic, symbolic, and real-world registers that mainly emerged during their attempts to illustrate or justify their reasoning and arguments on the explored phenomenon. It seems that AR was utilized as an instrument to transfer signs, including after the students removed the AR headset. The students continued to share their insights regarding what they visualized. This suggests that AR affordances, which allow juxtaposing mathematical representations with real-world phenomena, enable transferring signs, such as a mass-length graph, or a table of values, to become instruments for the students [27]. Such a transfer of signs opens opportunities to shift among representations. Therefore AR, which allows interaction with representations, such as virtual objects [28–29], seems to possess the potential to shift between representations, which is crucial for mathematical reasoning and covariational reasoning in our context. This is in line with literature on digital technologies that allow moving between different representations, thus supporting students' linking and understanding of the relations between representations (e.g., [30–31]).

We can summarize the set of the design principles derived from these two cycles of our AR learning environment for covariational reasoning as follows:

T1: Multiple representations promote engagement in covariational reasoning.
T2: Compatibility between the explored phenomenon and the representation is essential to engage students in covariational reasoning.
T3: The technology design should consider the following aspects: appropriate representations, data display, and mirroring.
T4: Juxtaposing virtual objects with real-world dynamic phenomena promotes understanding of the phenomenon.
T5: Real-time data provided on dynamic situations supports engagement in covariational reasoning.
T6: Effective use of AR technology entails pre-practice.

6.4 Future design considerations for the third cycle

Following the second cycle, no new significant technological problems were identified during the learning process. The AR technology functioned properly and the technological design considerations, as previously detailed, supported the students' engagement in covariational reasoning. However, for the third cycle, we will consider examining some minor technological aspects that allow for further elaborations of the presentation of the actual phenomena, for example, technological additions allowing motion representations at varying speeds may give added value to the design. In addition, we will stress a significant non-technological design aspect concerning the role of the teacher in the learning process.

During the second cycle, we noticed that after experimenting, students sometimes exchanged their AR headsets to share and explain their insights to each other. The students' discussion basically referred to the graphical and numerical representations, which turned to static mode, after experimenting. To create better opportunities for sharing the phenomenon and mathematical representations in a dynamic mode of such situations, we suggest including a slow-motion option of the graphical and numerical representations and perhaps a fast-motion. For it is often the case that a pattern can be seen more clearly if it is slowed down, or sometimes speeded up, depending on the pace of the actual phenomenon. Giving students the opportunity to replay and observe again the real dynamic phenomenon and representations at different rates may sharpen their reflections and support their discussion. After all, Galileo, for example, originally employed the inclined plane in "the Galileo experiment" precisely to slow down the motion of a falling body!

Although the role of the teacher was limited along the first and second cycles, we noticed that some of students' covariation statements, including the advanced levels, emerged following the researcher's questions which aimed to clarify students' thoughts. For the third cycle, we need to examine the use of such technology in a classroom setting, where the teacher may play a significant role in the learning process with a large group of students. Adapting the AR learning environment to an ordinary classroom may itself require some technological innovations; for example, we need to consider how the social interactions, mentioned above, can be extended by means of simultaneous projectors to an entire class discussion instead of a very small group of students working with one or two headsets. More importantly, though, teachers will have to learn to coordinate such AR sessions in order best to support the students' covariational reasoning.

As mentioned above, the way an AR learning environment leads students to connect mathematical representations with observed phenomena, and thus ultimately engage them in different levels of covariational reasoning, is dependent not just on the technology but on the compatibility of the phenomena and the various representations afforded by AR. Only two kinds of phenomena were studied here, one linear in character and one quadratic. This suggests the need to explore other instances of dynamic phenomena that may connect more naturally and compellingly to other instances of covariation and their mathematical representations. This involves the mutual development of task designs and further affordances of AR to serve their realization.

7 Acknowledgements

The research reported in this paper was funded by the Israel Science Foundation (Grants No. 1089/18).

8 References

[1] P. W. Thompson and M. P. Carlson, "Variation, covariation, and functions: Foundational ways of thinking mathematically," *Compendium for Research in Mathematics Education*, pp. 421–456.

[2] M. Carlson, S. Jacobs, E. Coe, S. Larsen, and E. Hsu, "Applying covariational reasoning while modeling dynamic events: A framework and a study," *Journal for Research in Mathematics Education*, vol. 33, no. 5, p. 352, 2002. https://doi.org/10.2307/4149958

[3] P. W. Thompson, N. J. Hatfield, H. Yoon, S. Joshua, and C. Byerley, "Covariational reasoning among U.S. and South Korean secondary mathematics teachers," *The Journal of Mathematical Behavior*, vol. 48, pp. 95–111, 2017. https://doi.org/10.1016/j.jmathb.2017.08.001

[4] T. Paoletti and K. C. Moore, "The parametric nature of two students' covariational reasoning," *The Journal of Mathematical Behavior*, vol. 48, pp. 137–151, 2017. https://doi.org/10.1016/j.jmathb.2017.08.003

[5] A. S. Zieffler and J. B. Garfield, "Modeling the growth of students' covariational reasoning during an introductory statistics course," *Stat. Educ. Res. J.*, vol. 8, no. 1, pp. 7–31, 2022. https://doi.org/10.52041/serj.v8i1.455

[6] M. Doorman, P. Drijvers, K. Gravemeijer, P. Boon, and H. Reed, "Tool use and the development of the function concept: From repeated calculations to functional thinking," *International Journal of Science and Mathematics Education*, vol. 10, no. 6, pp. 1243–1267, 2012. https://doi.org/10.1007/s10763-012-9329-0

[7] A. Hoffkamp, "The use of interactive visualizations to foster the understanding of concepts of calculus: Design principles and empirical results," *ZDM*, vol. 43, no. 3, pp. 359–372, 2011. https://doi.org/10.1007/s11858-011-0322-9

[8] R. Duval, "A cognitive analysis of problems of comprehension in a learning of mathematics," *Educational Studies in Mathematics*, vol. 61, no. 1–2, pp. 103–131, 2006. https://doi.org/10.1007/s10649-006-0400-z
[9] O. Swidan, F. Schacht, C. Sabena, M. Fried, J. El-Sana, and F. Arzarello, "Engaging students in covariational reasoning within an augmented reality environment," *Augmented Reality in Educational Settings*, pp. 147–167, 2019. https://doi.org/10.1163/9789004408845_007
[10] A. Watson, M. Ayalon, and S. Lerman, "Comparison of students' understanding of functions in classes following English and Israeli national curricula," *Educational Studies in Mathematics*, vol. 97, no. 3, pp. 255–272, 2017. https://doi.org/10.1007/s10649-017-9798-8
[11] Y. Levy, Y. O. Jaber, O. Swidan, and F. Schacht, "Learning the function concept in an augmented reality-rich environment," *Mathematics Education in the Digital Age (MEDA)*, 239.
[12] J.-C. Yen, C.-H. Tsai, and M. Wu, "Augmented reality in the higher education: students' science concept learning and academic achievement in astronomy," *Procedia—Social and Behavioral Sciences*, vol. 103, pp. 165–173, 2013. https://doi.org/10.1016/j.sbspro.2013.10.322
[13] I. Irwanto, R. Dianawati, and I. R. Lukman, "Trends of augmented reality applications in science education," *International Journal of Emerging Technologies in Learning (iJET)*, vol. 17, no. 13, pp. 157–175, 2022. https://doi.org/10.3991/ijet.v17i13.30587
[14] H.-K. Wu, S. W.-Y. Lee, H.-Y. Chang, and J.-C. Liang, "Current status, opportunities and challenges of augmented reality in education," *Computers & Education*, vol. 62, pp. 41–49, 2013. https://doi.org/10.1016/j.compedu.2012.10.024
[15] J. Ferrer-Torregrosa, J. Torralba, M. A. Jimenez, S. García, and J. M. Barcia, "ARBOOK: Development and assessment of a tool based on augmented reality for anatomy," *Journal of Science Education and Technology*, vol. 24, no. 1, pp. 119–124, 2014. https://doi.org/10.1007/s10956-014-9526-4
[16] I. Radu, "Augmented reality in education: A meta-review and cross-media analysis," *Personal and Ubiquitous Computing*, vol. 18, no. 6, pp. 1533–1543, 2014. https://doi.org/10.1007/s00779-013-0747-y
[17] Sudirman, et al., "Investigating the potential of integrating augmented reality into the 6E instructional 3D geometry model in fostering students' 3D geometric thinking processes," *International Journal of Interactive Mobile Technologies (IJIM)*, vol. 16, no. 06, pp. 61–80, 2022. https://doi.org/10.3991/ijim.v16i06.27819
[18] R. Duval, "Geometrical pictures: Kinds of representation and specific processings," *NATO ASI Series*, pp. 142–157, 1995. https://doi.org/10.1007/978-3-642-57771-0_10
[19] A. Bakker and D. van Eerde, "An introduction to design-based research with an example from statistics education," *Advances in Mathematics Education*, pp. 429–466, 2014. https://doi.org/10.1007/978-94-017-9181-6_16
[20] M. Q. Patton, *Qualitative Research and Evaluation Methods*, 3rd ed. Thousand Oaks, Calif.: Sage Publications, 2002.
[21] J.-E. Lee and M. Y. Lee, "How elementary prospective teachers use three fraction models: Their perceptions and difficulties," *Journal of Mathematics Teacher Education*, 2022. https://doi.org/10.1007/s10857-022-09537-4
[22] J. Bacca, R. Fabregat, S. Baldiris, S. Graf, and Kinshuk, "Augmented reality trends in education: A systematic review of research and applications," *Educ Technol Soc*, vol. 17, pp. 133–149, 2014a.
[23] C.-M. Chen and Y.-N. Tsai, "Interactive augmented reality system for enhancing library instruction in elementary schools." *Computers & Education*, vol. 59, no. 2, pp. 638–652, 2012. https://doi.org/10.1016/j.compedu.2012.03.001

[24] K. R. Bujak, I. Radu, R. Catrambone, B. MacIntyre, R. Zheng, and G. Golubski, "A psychological perspective on augmented reality in the mathematics classroom," *Computers & Education*, vol. 68, pp. 536–544, 2013. https://doi.org/10.1016/j.compedu.2013.02.017

[25] W. Nielsen, A. Turney, H. Georgiou, and P. Jones, "Meaning making with multiple representations: A case study of a preservice teacher creating a digital explanation," *Research in Science Education*, vol. 52, no. 3, pp. 871–890, 2022. https://doi.org/10.1007/s11165-021-10038-2

[26] J. Cai, "U.S. and Chinese teachers' constructing, knowing, and evaluating representations to teach mathematics," *Mathematical Thinking and Learning*, vol. 7, no. 2, pp. 135–169, 2005. https://doi.org/10.1207/s15327833mtl0702_3

[27] L. Trouche, "From artifact to instrument: Mathematics teaching mediated by symbolic calculators," *Interacting with Computers*, vol. 15, no. 6, pp. 783–800, 2003. https://doi.org/10.1016/j.intcom.2003.09.004

[28] R. Azuma, Y. Baillot, R. Behringer, S. Feiner, S. Julier, and B. MacIntyre, "Recent advances in augmented reality," *IEEE Computer Graphics and Applications*, vol. 21, no. 6, pp. 34–47, 2001. https://doi.org/10.1109/38.963459

[29] R. Capone and M. Lepore, "Augmented reality to increase interaction and participation: A case study of undergraduate students in mathematics class," in *Lecture Notes in Computer Science*, Cham: Springer International Publishing, 2020, pp. 185–204. https://doi.org/10.1007/978-3-030-58468-9_15

[30] J.-B. Lagrange, "New representational infrastructures: Broadening the focus on functions," *Teaching Mathematics and its Applications*, vol. 33, no. 3, pp. 179–192, 2014. https://doi.org/10.1093/teamat/hru018

[31] M. A. Navarro and P. Carreras, "An empirical study in convergence via iteration and its visualization," *Teaching of Mathematics*, vol. 15, no. 2, pp. 85–112, 2012.

9 Authors

Otman Jaber is a Ph.D. student at Ben-Gurion University of the Negev. His main research interest focuses on investigating the potential of the augmented-reality (AR) technology in learning settings, especially for developing mathematical concepts. E-mail: jaberot@post.bgu.ac.il.

Osama Swidan is a senior lecturer at the Ben-Gurion University of the Negev. His main research interests are concerned with integrating and designing new technologies, such as augmented reality, digital textbooks, and dynamic digital technologies, in learning and teaching mathematics at secondary levels. E-mail: osamas@bgu.ac.il.

Michael N. Fried is an associate professor of mathematics education at Ben Gurion University of the Negev. His research interests are eclectic and include mathematics pedagogy, teacher education, sociocultural issues, semiotics, and history of mathematics, as well as uses of technology such as augmented-reality in mathematical teaching and learning. E-mail: mfried@bgu.ac.il.

Article submitted 2023-02-16. Resubmitted 2023-03-23. Final acceptance 2023-03-23. Final version published as submitted by the authors.