Model of Carriage Movement on Plane with Dry Friction Forces

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Abstract—Wheel slip model is an important aspect of vehicle driving stability and safety control. However, in most of the carriage movement models friction forces and wheel slip effect are being neglected. This paper raises the problem of wheel slip in dangerous driving and autonomous vehicles under critical driving modes, then tasks the modeling of movement of an individual vehicle is considered. These are two tasks of nonholonomic mechanics: 1) the movement of a wheelset without wheel slip (two disks freely mounted on an axis) along an inclined plane in the field of gravity, and 2) the movement of a flat wheel model, which, under certain assumptions, can be a four-wheeled carriage. In these tasks, in addition to the standard dynamics (continuous motion without wheel slip), critical situations can also be observed associated with wheel sleep and separation of wheels from plane.

Keywords—Wheel slip, automobile, traffic accident, dangerous driving, simulation, dry friction, nonholonomic mechanics, carriage model.

1 Introduction

1.1 Dangerous driving

The steady increase in motorization has made personal vehicles widely available. The number of cars in the streets is constantly increasing. Due to the fact that the accident rate of road transport causes enormous physical and financial damage to society in general and citizens in particular, the problem of transport safety is one of the priority tasks for any government [1]. This problem is especially acute in large cities where saturated complex transport networks are overloaded and population density is up to

maximum limit [2]. There are many reasons why traffic accidents occur. Among them are natural disasters, destruction of the roadway, slippery road surface, failure of traffic lights, ignoring road signs and markings, inattention of road users to the situation on the road and to other road users, state of alcohol or drug affect, strokes and sudden health problems of drivers, as well as an unbalanced mental state of mind or a state of panic. Since the road transport network is a complex system, in order to maintain normal functioning, it requires constant involvement and attention from all its participants [3]. Therefore, any neglect of this complex system in favor of driver's personal interests can cause fatal consequences.

More and more the situation of the constant use of mobile phones while driving is used as an example of such attitude. However, another, no less frequently occurring side of the problem is aggressive driving or driving in a state of panic. Panic and aggressiveness from the psychological point of view lead to the fact when a person should not feel like a member of a group, feel fear for his own life, and begins to act logically, but not in the laws and not according to the rules of groups, but in his own individual manner, violating the coherence of the whole system [4]. In this state, the driver can make decisions that are unpredictable for other participants, so they may either not have time to react, or these actions will provoke emergency or panic actions, which will lead to a dangerous situation or to a collision. This phenomenon has various names: dangerous driving, aggressive driving, careless driving and reckless driving. Laws of many countries provide for various fines and other penalties for such actions. The Government of the Russian Federation identifies the following types of dangerous driving [5]:

- Multiple and sudden lane changes
- Lane change with small gaps (headways)
- Tailgating (decrease of headways)
- Decreasing lateral interval
- Emergency breaking and rubber-necking
- Obstructing overtaking

Nevertheless, despite the apparent clarity of the classification of dangerous driving, there are many questions from the law point of view. As noted on the official website of the Road Traffic Organization of the Russian Federation, "the line between creating a threat is very thin, subjective, and uncertain" [6]. Therefore, in most cases it becomes possible to qualify dangerous driving only after accident occurred. And the punishment for dangerous driving, if it is proved to be the cause of the accident, is additional to the main consequences for the immediate actual accident.

The attempts to formulate the qualitative characteristics of such type of dangerous driving as "Multiple and sudden lane changes" by analyzing the vehicle's trajectory along GPS coordinates were made by our research team in the article [7].

1.2 Autonomous transport

The high injury risk of road transport caused by the instability of human psychology can be eliminated by increasing the level of autonomy of the car. Most specialists believe that the exclusion of the human factor from the transport system will allow reducing accident rate and optimizing the efficiency of transport networks.

Self-driving cars, or autonomous vehicles (AVs), connected and autonomous vehicles (CAVs), driverless cars, robo-cars or robotic cars, development of which begin history back in the 20s of the last century, have become an obvious trend of our time, and plans for the development and implementation of autonomous vehicles have become part of the national programs of many countries, automotive manufacturing companies, as well as institutes and research centers. Currently, such large projects in this area are known as: Autopilot of Tesla, Cruise of GM, Zoox (California), Nuro (California), Pony.AI (China, Guangzhou), VW, Audi, Symbioz by Renault, BMW, Volvo, Nissan, Waymo by Google, Robo-Taxi by Yandex, Baidu, Almotive (Hungary), AutoX Technologies (Hong Kong and San Jose), WeRide (China, Guangzhou), AKTIV, VisLab, Leonie from Braunschweig, KAMAZ with Cognitive Technologies and others. Also in this regard, one cannot fail to mention such programs for the development of an autonomous vehicle as the European Commission Program, the 2getthere program in the Netherlands, the ARGO program in Italy, and the DARPA Grand Challenge in the USA.

SAE International has established 6 levels of autonomy for self-driving cars (See Fig.1). These levels from zero to fifth characterize the degree of control of artificial intelligence over the vehicle and the degree of human involvement in the process of driving. Level zero is a car without any automation; ABS and ordinary cruise control do not count. Level 1 includes cars with steering and braking control, adaptive cruise control. Level 2 cars can control taxiing and braking in certain areas, such as highways, but the driver must still be vigilant and be ready to take control at any time. Level 3 allows the driver not to watch the headway all the time. Level 4 is already a full autopilot, but it only works in certain conditions and in a marked area, it even allows the driver to take a nap. Level 5 is a complete autopilot without any restrictions. No steering wheel is required at this level [8].
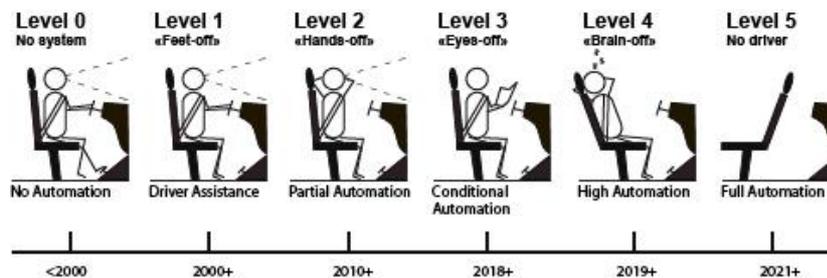

Fig. 1. Automation levels

The implementation of the autonomous vehicle system includes the following hardware elements: cameras and optical instruments, system-on-chip (SoC), radar, a communication system with other vehicles and infrastructure (V2V/V2I), active remote sensing system LiDAR (Light Detection and Ranging), converters of digital signal to mechanical actions, ultrasonic sensors, GPS, odometric sensors, etc., as well as software elements such as: high-resolution maps, methods of simultaneous localization and map building (SLAM, simultaneous localization and mapping), algorithms for data analysis and processing, recognition of obstacles and objects, prediction of trajectories and actions of moving objects, decision-making algorithms, such as planning routes, making maneuvers, accelerating, braking, taxiing, an operating system for driving a car, a supervisory control system for identifying and eliminating errors, as well as a system for interacting with an external data collection center and cloud services [9].

The development of robotics, machine vision, artificial intelligence, increasing the speed of data processing made it possible to implement autonomous vehicle. But the new concept of transport systems without the active participation of the person who delegated the right to drive to autonomous systems, together with solving a number of traditional issues, brings an even greater number of new problems. These include problems of liability in the case of damage, loss of the ability to drive a vehicle independently, reliability and vulnerability of software, loss of privacy, the danger of using autonomous cars for terrorist activities, the ethical issue of minimizing casualties when making decisions in critical situations etc. [10].

Another important aspect of driving in extreme situations is the study of the mechanical characteristics of the adhesion of the wheels of a moving car to the roadway. The fact is that when performing a speed maneuver while decreasing the distance between the cars, the driver makes decisions based on his experience, may not take into account all conditions, such as abrasion of treads, defects in the road surface, various degrees of moisture in the roadway, etc. This problem is relevant both for those cases when a person is driving a car, and for improving the control algorithms of autonomous vehicles.

2 Description of the Wheelset Model

2.1 Description of the wheelset system without wheel slip

Consider a wheelset on an inclined plane, shown in Figure 2. This design consists of two homogeneous disks, each with mass m and radius b, freely mounted on the axis O_1O_2, which is a uniform axle of mass M and length $2l$. Thus, disks (wheels) can freely and independently rotate around the axis O_1O_2 without friction.
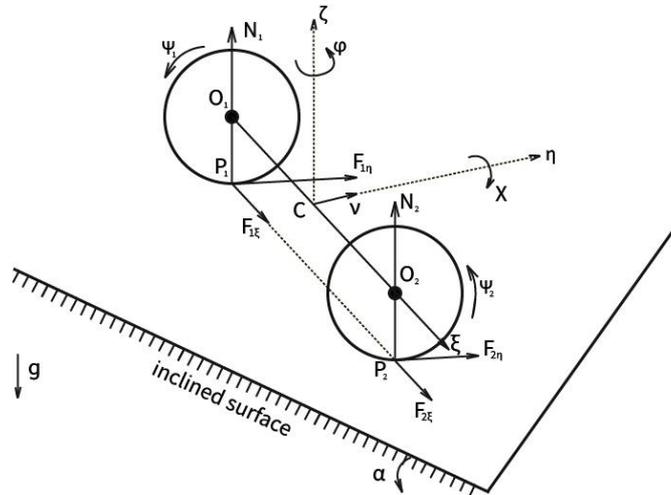
Fig. 2. Wheelset on surface

The described system moves under the influence of gravity along a rough inclined plane (the angle of inclination to the horizontal is *a*) so that at the points of contact of the wheels with the plane (points P_1, P_2, P1, P2, see Fig. 2), and no wheel slip (neither lateral nor longitudinal) occurs.

2.2 Basic kinematic variables

Let C — be the center of mass of the system (it is clear that it is in the middle of the segment O_1O_2), and $C\xi\eta\zeta$ — is the coordinate system rigidly connected with the axis O_1O_2, moreover $C\xi$ is directed along the axis O_1O_2, $C\zeta$ — directed upward along the normally inclined plane, and $C\eta$ — is perpendicular to $C\xi$ and $C\eta$ and forms the right coordinate system with them.

The movement of the wheelset occurs under the influence of gravity and reaction forces of the plane that occur at the contact points P_1 and P_2. These forces are shown in fig. 2: N_1, N_2 — normal reaction forces, and the values of their projections onto the axis $C\xi$ should be non-negative ($N_{1\zeta} \geqslant 0$, $N_{2\zeta} \geqslant 0$), $F_{1\xi}$, $F_{2\xi}$ — lateral, and $F_{1\eta}$, $F_{2\eta}$ — longitudinal forces adhesion (static friction forces). In the future, we will assume that separation does not occur and $F_{1\eta}$, $N_{2\zeta}$ are substantially positive values. In addition, let the friction coefficient of the reference plane be so large that there is no wheel slip at points P_1, P_2, thus $V_{P1} = V_{P2} = 0$.

Let ψ_1, ψ_2 be the proper angles of rotation of the wheels relative to the axis O_1O_2, φ is the angle of rotation of the axis O_1O_2 in an inclined plane, counted counterclockwise from the line of the largest slope (if viewed from the positive end of the axis $C\xi$).

Thus, the angular velocity of the axis O_1O_2 (and hence the of system $C\xi\eta\zeta$) in the projections on the axis ξ, η, ζ is:

$$w_0 = (0, 0, j) \tag{1.1}$$

The angular velocity of each of the wheels, in projections on the same axis:

$$w_j = (\dot{y}_j, 0, \ddot{j}), \quad j=1.2 \tag{1.2}$$

Let's find the speed of point O1, considering it to belong to the wheel 1:

$$n_{O_1} = n_{P_1} + [w_1 \times P_1 O_1] \tag{1.3}$$

Hence, since $n_{P_1} = 0$, we obtain:

$$u_{O_1 x} = 0, \ u_{O_1 h} = -b\dot{y}_1, \quad u_{O_1 z} = 0 \tag{1.4}$$

On the other hand, since the point O1 belongs to the axis O1O2, we obtain:

$$n_{O_1} = n_C + [w_1 \times CO_1]$$

or, in projections on the axis ξ, η, ζ we obtain:

$$u_{O_1 x} = u_{Cx}, \quad u_{O_1 h} = u_{Ch} - lj, \quad u_{O_1 z} = u_{Cz}$$

From (1.3), (1.4) it follows

$$u_{Cx} = u_{Cz} = 0, \quad u_{Ch} - lj = -b\dot{y}_1$$

Similar logic for the point O$_2$ leads to the relations:

$$u_{Cx} = u_{Cz} = 0, \quad u_{Ch} - lj = -b\dot{y}_2$$

Thus, no wheel slip condition leads to the following conclusions:

1. The speed of point C has a nonzero projection only onto the longitudinal axis Ch, we will denote it hereafter by u, which is

$$u_{Cx} = 0, \ u_{Ch} = u, \ u_{Cz} = 0 \tag{1.5}$$

2. Two relations are satisfied:

$$\begin{cases} u + lj = -b\dot{y}_2 \\ u - lj = -b\dot{y}_1 \end{cases} \tag{1.6}$$

2.3   The equation of motion of wheelset without wheel slip

The equations of motion of the center of mass of the system in the axes Cxhz, taking into account (1.1) and (1.4), we obtain:

$$\begin{cases} m_0(-vj) = F_1 x + F_2 x + m_0 g \sin\alpha \sin j, \\ m_0 \dot{v} = F_1 h + F_2 h + m_0 g \sin\alpha \cos j, \\ 0 = N_1 + N_2 - m_0 g \cos\alpha, \end{cases} \tag{1.7}$$

Where $m_0 = M + 2m$ is the aggregate mass of the system.

Let's denote J_ξ, J_z be the axial and equatorial moments of inertia of the wheel ($J_\xi = mb^2/2, J_z = mb^2/4$), respectively, J_1 is the mass moment of inertia of the axle relative to axis Cz(or Ch) ($J_1 = Ml^2/3$). Using the symmetry of the system and formulas (1.1) and (1.2) for the angular velocities, we obtain that the kinetic moment is follows:

$$K_{C\xi} = J_\xi(\dot{y}_1 + \dot{y}_2), \ K_{Ch} = 0, \ K_{Cz} = J_0\dot{j},$$ (1.8)

Where $J_0 = J_1 + 2J_z + 2ml^2$. Using (1.8) and relation (1.1) for the angular velocity Cxhz, we obtain the equations of the theorem on the change in the kinetic momentum K_C in the Cxhz axes:

$$\begin{cases} J_\xi(\ddot{y}_1 + \ddot{y}_2) = b(F_{1h} + F_{2h}), \\ J_\xi(\dot{y}_1 + \dot{y}_2) = (N_1 - N_2)l - b(F_{1\xi} + F_{2\xi}), \\ \qquad J_0\ddot{j} = l(F_{2h} - F_{1h}). \end{cases}$$ (1.9)

Now, we write down the equations for changing the kinetic moment for each of the wheels separately with respect to the ξ axis, under the assumption that they rotate freely on the ξ axis and there are no rotations around Ch axis, i.e.:

$$J_\xi\ddot{y}_j = bF_{jh}, j = 1,2.$$ (1.10)

Thus, we obtain a closed system of 10 equations (1.6)-(1.10) for determining 10 variables: $v, j, y_1, y_2, N_1, N_2, F_{1\xi}, F_{2\xi}, F_{1h}, F_{2h}$.

2.4 Analysis of the system of equations (1.6)-(1.10)

Let the initial conditions be:

$$\{v_C(0), j(0), j(0), \dot{y}_1(0), \dot{y}_2(0)\}$$

such that equalities (1.5) and (1.6) are satisfied at t=0 (the moment of start of the wheelset movement in the plane, so that the kinematic coupling conditions are not violated). It is required to determine the solution of system (1.6)-(1.10), which satisfies the continuity condition: $N_j > 0, j = 1,2$. In a more accurate statement, if the friction coefficient f of the plane with wheels is specified, it is also necessary to satisfy Coulomb's inequalities in the process of motion:

$$F_{j\xi}^2 + F_{jh}^2 \le f^2 N_j^2, j = 1,2.$$

This kind of problem was considered for the Chapylygin sleigh on an inclined plane with friction in work [11].

As shown in [12], the solution has the form:

$$v(t) = v_0 + \frac{g\sin a}{\omega_0(1 + \frac{2p^2}{b^2})}[\sin(\omega_0 t + \varphi_0) - \sin\varphi_0].$$ (1.11)

Let x be the coordinate of the center of mass C along the line of the largest slope (down the inclined plane), y – be the coordinate perpendicular to it, then we have:

$$\dot{x} = v\cos\varphi, \dot{y} = v\sin\varphi.$$

Integration of the last equalities leads to the conclusion that the center of mass of motion along the cycloid perpendicular to the line of the largest slope, that is, the wheel-set does not roll down.

2.5 Critical situations

Example:

$$\frac{\omega_0 v_0}{g}\left(1 + \frac{2p^2}{b^2}\right) < \frac{1}{b}\cos a - 2\sin a + \sin a \sin\varphi_0. \tag{1.12}$$

For example, for $v_0 = 0$, $\varphi_0 = 0$ we have $tga < l/(2b)$, that is, a rollover can happen for sufficiently large a: $tga > l/(2b)$.

If inequality (1.15) is violated, then there will definitely come a point in time that a normal reaction is zero, for example, N_1. So, let $N_1(t_1) = 0$ and further movement is possible only with separation of wheel 1 (while maintaining the contact of wheel 2). Consider the beginning of such a separation, that is, the resulting rotation of the wheel-set around the axis Ch. We will determine such a rotation by the angle x, count it counterclockwise, if we look from the positive part of the axis Ch (see Fig. 2). Considering the further movement, we assume that $x \approx 0, \dot{x} \approx 0, \ddot{x} \neq 0$. Then we have:

$$v_{CJ\xi} = 0, v_{Ch} = v, v_{Cz} = l\dot{x};$$

$$K_{CJ\xi} = J_\xi(\dot{y}_1 + \dot{y}_2), K_{Ch} = J_0\dot{x}, K_{Cz} = J_0,$$

$$\omega_0 = (0, \dot{x}, \dot{\varphi}).$$

After several operations we obtain:

$$\left(\frac{J_0 + m_0 l^2}{m_0}\right)\ddot{x} = \dot{v}\dot{\varphi}\left(b + \frac{2p^2}{b}\right) + v\ddot{\varphi}\left(b + \frac{2p^2}{b}\right) + gb\sin a\cos\varphi\dot{\varphi} =$$

$$= gb\sin a\cos\varphi\left[\left(1 + \frac{2p^2}{b}\right)\left(\dot{\varphi}\frac{p_0^2 b^2 + p^2 l^2}{p^2 p_0^2 + p_0^2 b^2 + p^2 l^2} - v\frac{p^2 l}{p^2 p_0^2 + p_0^2 b^2 + p^2 l^2}\right) + \dot{\varphi}\right]$$

$$=$$

$$= gb\sin a\cos\varphi\left[\left(1 + \frac{2p^2}{b}\right)\frac{\dot{\varphi}(p_0^2 b^2 + p^2 l^2) - vp^2 l}{p^2 p_0^2 + p_0^2 b^2 + p^2 l^2} + \dot{\varphi}\right], \tag{1.13}$$

where $p^2 = \frac{J_\xi}{m_0}, p_0^2 = \frac{J_0}{m_0}$.

It is clear that choosing ω_0 small enough, we can achieve large $v(t_1)$, and formula (1.13) then gives us $\ddot{x} < 0$, that is, a paradoxical situation arises when the separation occurs. These considerations lead us to the conclusion that, in cases where there can be a zeroing of the normal reaction, it is necessary to take into account the *finite coefficient*

of friction and consider the process of movement of the wheelset with possible transitions to the situation of wheel slippage, which, of course, complicates the solution of the problem.

3 Description of the Carriage Model

3.1 Model description

Figure 3 shows a flat wheel model of two wheelsets A_1A_2, B_1B_2, connected at their centers of mass by a rigid axle AB. The joints in connection points *A* and *B* are assumed to be ideal and cylindrical, and the rolling of wheels A_1, A_2, B_1, B_2 along the supporting plane occurs without slipping, i.e. a nonholonomic connection is imposed.

In this model, we assume (same as in [1-4]), that this coupling is realized only by the *transverse forces* F_A and F_B, applied to A_1A_2 and B_1B_2, respectively (see Fig. 3). The consideration of longitudinal forces was considered, for example, in paper [7].

The presence of a non-holonomic connection gives the right to assume that the velocities of the centers of mass of the wheelsets, denoted as v_A and v_B, are always directed *perpendicular* to the corresponding axes, i.e. $v_A \perp A_1A_2$, $v_B \perp B_1B_2$ (see Fig.3).

3.2 Description of symbols

Further, the following notation is introduced: θ_1, θ_2 – the deflection angles of the wheelsets A_1, A_2, B_1, B_2 from the longitudinal axis ξ of the flat wheel model, and θ_1 is counted clockwise, and, θ_2 is counterclockwise; $AC = a, CB = b$, *C* is the center of mass of the entire system, which we assume to be unchanged, that $a = const, b = const$; φ is the angle of rotation of the axle AB relative to the invariably oriented axis C_x, counted counterclockwise; P is the instantaneous velocity center of the axle AB, which, due to nonholonomic constraints, is the intersection of lines A_1A_2 and B_1B_2; m_{AB}, m_A, m_B are the masses of axle AB, of wheelsets A_1, A_2, B_1, B_2; J_{AB} is the moment of inertia of the axle AB together with the concentrated masses $m_A \ and \ m_B$ at its ends relative to point C; J_A, J_B are the moments of inertia of the wheel pairs, respectively A_1, A_2, B_1, B_2 relative to the own centers of mass A and B; $C_{\xi h}$ is the coordinate system rigidly connected with the axle AB.
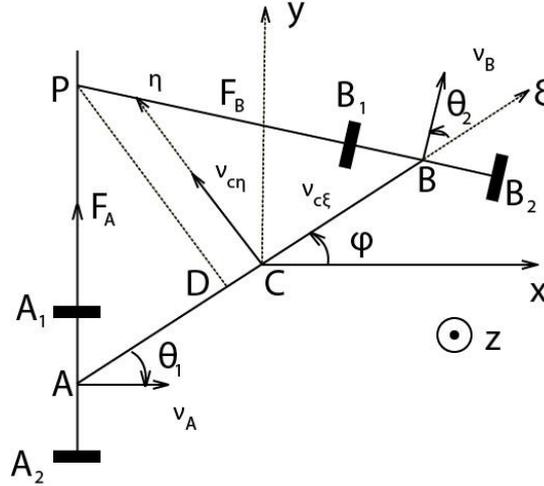
Fig. 3. Carriage on surface

3.3 Equations of motion

For the three angles $\varphi, \theta_1, \theta_2$ we can write three equations of dynamics that do not contain unknown forces and reaction forces in the hinges A and B:

$$K_{Pz} + m[v_p \times v_c]_z = 0, \tag{2.1}$$

$$J_A(\ddot{\varphi} - \ddot{\theta}_1) = M_{Az} \tag{2.2}$$

$$J_B(\ddot{\varphi} + \ddot{\theta}_2) = M_{Bz} \tag{2.3}$$

Equation (2.1) is the equation of the kinetic moment relative to the moving point P; and equations (2.2) and (2.3) are the equations of kinetic moments of wheel sets relative to their own centers of mass. The following notation is introduced in these equations: $m = m_{AB} + m_A + m_B$ is the total mass of the system; K_{Pz} is the kinetic moment of the system relative to the point P, v_p is the speed of the point P (as geometric relative to the motionless coordinate system), v_c is the velocity of the center of mass C; M_{Az}, M_{Bz} control moments in joints A and B (if there are any). Let D be the projection of the point P onto the axle AB (see Fig 3). Denote $CD = \xi_0$. Then we have, in accordance with Koenig's theorem:

$$K_{Pz} = K_{cz} + m(v_{c\xi} \cdot PD + v_{ch} \cdot CD)$$

$$K_{cz} = J_{AB\dot{\varphi}} + J_B(j + \dot{\theta}_2) + J_A(j - \dot{\theta}_1)$$

After several operations we obtain:

$$\ddot{\jmath}\left\{J_{AB} + \frac{m}{(tg\theta_1 + tg\theta_2)^2}[(a+b)^2 + (atg\theta_2 - btg\theta_1)^2]\right\} -$$

$$- \frac{m\dot{\varphi}(a+b)}{(tg\theta_1 + tg\theta_2)^3}\left[\frac{\theta_2}{cos^2\theta_2}(a+b - atg\theta_1 tg\theta_2 + btg^2\theta_1) +\right.$$

$$\left. + \frac{\theta_1}{cos^2\theta_1}(a+b + atg^2\theta_2 - btg\theta_1 tg\theta_2)\right] = -M_{Az} - M_{Bz}. \tag{2.4}$$

Equations (2.2) - (2.4) form a closed (with respect to $\varphi, \theta_1, \theta_2$) system of equations.

3.4   Case of the equation with fixed angles

We control the movement of the wheelset, that M_{Az} and M_{Bz} so that $\theta_1 = \theta_2 = 0$ (the same deviations of the wheelsets, but in different directions). Then from (2.2), (2.3) we get:

$$M_{Az} + M_{Bz} = (J_A + J_B)\ddot{\varphi}$$

And from equation (2.4) we get:

$$\ddot{\jmath}\left\{J_{AB} + J_A + J_B + \frac{m}{4tg^2\theta}[(a+b)^2 + tg^2\theta(a-b)^2]\right\} -$$

$$- \frac{m\dot{\varphi}(a+b)}{4tg^3\theta}\left[\frac{\theta}{cos^2\theta}(a+b)\right] = 0. \tag{2.5}$$

The energy integral takes the form of:

$$2T = \jmath^2\left[J_{AB} + \frac{(a-b)^2 + (atg\theta_2 + btg\theta_1)^2}{(tg\theta_1 + tg\theta_2)^2}\right] = h_0^2 = const. \tag{2.6}$$

3.5   Fixed wheelset

Consider a special case: a fixed wheelset A_1, A_2, that $\theta_1 = 0$, then the energy integral (2.6) has the form of:

$$\jmath^2\left[J_{AB} + m\frac{(a+b)^2 + a^2 tg^2\theta_2}{tg^2\theta_2}\right] = h_0^2 \ =>$$

$$\jmath^2[(J_{AB} + ma^2) + m(a+b)^2 ctg^2\theta_2] = h_0^2. \tag{2.7}$$

Equation (2.3) with $M_{Bz} = 0$ gives us:

$$\dot{\varphi} = \omega_0 - \dot{\theta}_2,$$

and from (2.7) we obtain $A^2 = J_{AB} + ma^2, B^2 = m(a+b)^2, \lambda = \frac{h_0}{\omega_0}, x = tg\theta_2$:

$$\frac{d\theta_2}{dt} = \frac{\omega_0\sqrt{B^2 + A^2 + tg^2\theta_2} - h_0 tg\theta_2}{\sqrt{B^2 + A^2 tg^2\theta_2}} \rightarrow$$

$$I = \int \frac{d\theta_2\sqrt{B^2 + A^2 tg^2\theta_2}}{\sqrt{B^2 + A^2 tg^2\theta_2} - \lambda tg\theta_2} = \omega_0 t + C$$

By denoting $x = tg\theta_2$, we obtain:

$$I = \int \frac{dx}{1+x^2} + \lambda \int \frac{xdx\sqrt{B^2 + A^2 tg^2\theta_2}}{(1+x^2)[B^2 + (A^2 - \lambda^2)x^2]} + \lambda^2 \int \frac{x^2 dx}{(1+x^2)[B^2 + (A^2 - \lambda^2)x^2]}.$$

It is easy to show that all these integrals are taken in elementary functions. Thus, in this case, the equations of motion of the system are integrated in time functions for *any parameter values and initial conditions*.

3.6 Controllable wheelset

$$\dot{\varphi}\left\{J_{AB} + J_A + J_B + \frac{m}{4}\left[\frac{(\dot{a}+b)^2}{tg^2\theta} + tg^2\theta(a-b)^2\right]\right\} - \frac{m\dot{\varphi}(a-b)^2}{4tg^2\theta} \cdot \frac{\dot{\theta}}{cos^2\theta} = 0;$$

$$\ddot{\varphi} = \frac{1}{2}\frac{d(\dot{\varphi}^2)}{d\varphi}; \quad \dot{\varphi}\dot{\theta} = (\dot{\varphi})^2\frac{d\theta}{d\varphi}; \quad A = \frac{m}{4}(a-b)^2 + J_{AB} + J_A + J_B;$$

$$d(\dot{\varphi}^2)\left[A + \frac{m(a+b)^2 ctg^2\theta}{4}\right] - \frac{m(a+b)^2}{2tg^3\theta cos^2\theta} \cdot (\dot{\varphi})^2 d\theta = 0 \rightarrow$$

$$\frac{d(\dot{\varphi}^2)}{\dot{\varphi}^2} = d\theta \cdot \frac{m(a+b)^2}{2tg^3\theta cos^2\theta\left[A + \frac{m(a+b)^2 ctg^2\theta}{4}\right]};$$

$$\dot{\varphi} = \frac{C_0 tg\theta}{\sqrt{Atg^2\theta + B}}.$$

The resulting equation is the main equation for the movement of a controlled wheelset.

4 Conclusion

In this paper we've found equation for the movement of a wheelset without wheel slip, where two wheels freely mounted on an axis and moving along an inclined plane in the field of gravity, another equation of movement of a wheel model, which, under certain assumptions, can be a four-wheeled carriage. In addition to dynamics of continuous motion without wheel slip, critical situations were observed associated with wheel

slip and separation of the carriage wheels from the plane. The related parameters were found and considered.

5 Acknowledgement

6 References

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