

PAPER

The Way to Construct Innovative Methods for Solving Initial-Value Problem of the Volterra Integro-Differential Equation

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ABSTRACT

Mathematical models for many problems in the natural sciences are often simplified to solving initial-value problems (IVPs) for the Volterra integro-differential equations (VIDE). Numerical methods of a multistep type are typically used to solve these problems. It is known that in some cases, the multi-step method (MSM) is applied to solving the IVPs of both ordinary differential equations (ODEs) and VIDE encountered in solving some problems in mathematical biology. Here, to solve such problems by combining different methods, some modifications of established methods were developed, and it was demonstrated that these methods outperform the existing ones. As is known, one of the main issues in solving the aforementioned problems is determining the reliability of calculating values using the known mathematical-statistical models (MSMs). In this regard, some experts utilize the predictor-corrector method. Having highlighted the disadvantages of this method, the proposal is to develop an innovative approach and assess the errors that may arise when applying this method to solve various problems. Here, the IVPs for the VIDE of the first order are primarily investigated. To illustrate the benefits of the innovative methods proposed here, we discuss the use of simple numerical methods to solve some common examples.

KEYWORDS

initial-value problem (IVP), ordinary differential equations (ODEs), Volterra integro-differential equation (VIDE), innovative method of multistep type (IMM), stability and degree, multi-step second derivative methods (MSDM)

1 INTRODUCTION

Renowned scholar Vito Volterra explored Earth's memory, population issues, specific epidemic challenges, and urgent matters where delay is not an option [1–6].

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Let the following task be given:

$$y' = f(x, y) + \delta_1 \int_{x_0}^x K(\delta_2 x, s, y(s)) ds, \quad y(x_0) = y_0, \quad x_0 < x < X_0. \tag{1}$$

In this study, we address the simple representation of IVPs for the Volterra integro-differential equations (VIDE) of the first order. This problem has been studied by several scientists [7–24]. In this context, we assume that this problem has a unique solution defined within a specified $[x_0, X]$ segment. We divide the segment $[x_0, X]$ into N sub-intervals by mesh points $x_{i+1} = x_i + h (i = 0, 1, \dots, N - 1)$ to find the numerical solution of problem (1). For the construction of numerical methods, it is assumed that the functions continue to a totality of arguments $f(x, y)$ and $K(x, s, y)$ are defined in a closed region where they have continued partial derivatives up to p , inclusively. Here, it is assumed that p -is the degree of use of the methods. Let us denote the approximate value of the solution of the problem (1) through y_i but the exact value as $y(x_i)$ ($i = 0, 1, \dots, N$). As is known, Volterra himself (1), considered the linear case when solving problem and aimed to find approximate values for the subsequent problem.

$$y'(x_k) = f(x_k, y(x_k)) + \delta_1 \int_{x_0}^{x_k} K(\delta_2 x_k, s, y(s)) ds, \tag{2}$$

used a formula that can be represented as:

$$y'(x_k) = f(x_k, y(x_k)) + \delta_1 h \sum_{i=0}^k \bar{\beta}_i K(\delta_2 x_k, x_i, y(x_i)) + R_k. \tag{3}$$

Here, R_n -is the reminder term. This equality at the mesh-point x_{k+1} can be presented as:

$$y'(x_{k+1}) = f(x_{k+1}, y(x_{k+1})) + \delta_1 h \sum_{i=0}^{k+1} \beta_i K(\delta_2 x_{k+1}, x_i, y(x_i)) + R_{k+1}. \tag{4}$$

By using the comparison of equations (3) and (4), we can see that at each step, the sum must be recalculated because it $K(x, s, y)$ depends on the mesh-point around which the calculation is being performed. Therefore, with an increase in the values of x , the amount of computation also increases. To address this limitation, some authors have developed a new approach that is not affected by this issue (refer to, for instance, [13–23]). We will use the k -step method, where the coefficients are constant, to solve equation (3). Then, we get the following:

$$\sum_{i=0}^k \alpha_i y_{n+i} = h \sum_{i=0}^k \beta_i f_{n+i} + h \delta_1 \sum_{i=0}^k \sum_{j=i}^k \gamma_i^{(j)} K(\delta_2 x_{n+j}, x_{n+i}, y_{n+i}), \quad n = 0, 1, \dots, N - k. \tag{5}$$

Noted that the condition $i \leq j \leq k$ depends on the properties of the kernel of the Volterra integral. If it $K(x, s, y(s))$ is independent from the x , ($K(x, s, y) = \varphi(s, y)$), then named inequality, it can be written as, $i \leq k$, which is obviously.

It is clear that all the properties of numerical methods depend on how the values of the coefficients $\beta_i, \gamma_i^{(j)}$ ($i, j = 0, 1, 2, \dots, k$) can be found. It follows that the choice of how to determine the values of the coefficients in method (5) is very important.

Depending on these methods, they are called k-step, multi-step, or, in other words, finite-difference methods, with constant coefficients.

2 METHOD FOR FINDING THE VALUES OF COEFFICIENTS IN THE EQUALITY OF (5)

It's obvious that in this case, $\delta_1 = 0$, from the equality of (5), it follows the well-known finite-difference method with constant coefficients. But if $\delta_1 = 1$ and $\delta_2 = 1$, then from the equality of (5):

$$\sum_{i=0}^k \alpha_i y_{n+i} = h \sum_{i=0}^k \beta_i f_{n+i} + h \sum_{i=0}^k \sum_{j=i}^k \gamma_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}), \quad n = 0, 1, \dots, N - k. \quad (6)$$

Equality (3) and (4) differ from equality (5) or (6) in that the equality of (5) and (6) latter use approximate values obtained after discarding the remainder term in the former (3) and (4).

It's obvious that the numerical methods of type (5) for this case $\delta_1 = 0$ are considered known if the values of its coefficients are given. In 1 general, the coefficients are selected so that the receiving method achieves the maximum order of accuracy. In this regard, let us consider finding the coefficients in method (5). In this case, Taylor series and the theory of systems of linear algebraic equations are typically used. In one version, the system of linear algebraic equations for this case $\delta_1 = 0$ can be presented as:

$$\begin{aligned} \sum_{i=0}^k \alpha_i &= 0; \quad \sum_{i=0}^k i \alpha_i = \sum_{i=0}^k \beta_i; \quad \sum_{i=0}^k \frac{i^2}{2!} \alpha_i = \sum_{i=0}^k i \beta_i \dots; \\ \sum_{i=0}^k \frac{i^p}{p!} \alpha_i &= \sum_{i=0}^k \frac{i^{p-1}}{(p-1)!} \beta_i. \end{aligned} \quad (7)$$

Separately, we will assume that in system (7), the sum of the equations will be equal to $p + 1$. In this case, we can prove the following lemma.

Lemma 1. If $\alpha_p, \beta_i (i = 0, 1, 2, \dots, k)$ are a solution to system (7), then $p_{max} = 2k$ and its coefficients, i.e., $\alpha_p, \beta_i (i = 0, 1, \dots, k)$ satisfying the following asymptotic equality:

$$\sum_{i=0}^k (\alpha_i y(x + ih) - h \beta_i y'(x + ih)) = O(h)^{p+1}, h \rightarrow 0.$$

It is obvious that in this system of algebraic equations, the sum of the equations is equal to $p + 1$, and the sum of the unknowns will be equal to $2k + 2$. To further study system (7), we calculate an integral of the form:

$$\mathcal{G}(x) = \int_{x_0}^x K(x, s, y(s)) ds. \quad (8)$$

For the calculation of the values, $\mathcal{G}(x)$, one can use the following references [1–8], [13], [14], [17], [22]:

$$\sum_{i=0}^k \alpha_i \mathcal{G}_{n+i} = h \sum_{i=0}^k \sum_{j=i}^k \gamma_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}), \quad n = 0, 1, \dots, N - k. \quad (9)$$

Theorem (Dahlquist). If method (11) has the degree of p , then there exist methods with degree $p \leq 2k$. Method with the degree $p_{max} = 2k$ is unique. If method (11) is stable, then $p \leq 2[k/2] + 2$. If method (11) is stable and $\beta_k = 0$ then $p \leq k$.

Thus, it has been shown how one can construct MMs with constant coefficients to solve problem (1). Note that to apply method (6) to solve problem (1), it is necessary to know the values of the solution at the first k points of problem (1). For finding the values, y_0 , one can use the initial value, and for calculating other initial values, one can use any one-step method. By using the methods described above, we can achieve more accurate results. However, some difficulties may arise during the application of these methods. Note that by overcoming these obstacles, it will be convenient to use predictor-corrector methods. It will be convenient to use the explicit method as a predictor method, and then one can continue the necessary calculations with the implicit method.

The famous scientist Dahlquist established that if equation (11) converges, then its coefficients must satisfy the corresponding conditions [12–14], [24–32]:

- A. $\alpha_i, \beta_i (i = 0, 1, \dots, k)$, which are coefficients, consist of real numbers and $\alpha_k \neq 0$.
- B. The following polynomials:

$$\rho(\lambda) \equiv \sum_{i=0}^k \alpha_i \lambda^i; \quad \delta(\lambda) \equiv \sum_{i=0}^k \beta_i \lambda^i.$$

Two polynomials do not have any common factors other than a constant.

- C. $\delta(1) \neq 0$ and the condition $p \leq 1$ are satisfied.

To construct a model with high accuracy, consider a MM of the form:

$$\sum_{i=0}^{k-m} \bar{\alpha}_i y_{n+i} = h \sum_{i=0}^k \bar{\beta}_i f_{n+i}; \quad n = 0, 1, \dots, N - k + m \quad (m > 0). \quad (14)$$

This method is usually referred to as the advanced multistep method.

V. Ibrahimov proved that if method (14) is convergent, then the coefficients will satisfy the corresponding conditions [12–19]:

- A. $\bar{\alpha}_j (j = 0, 1, \dots, k - m)$ and $\bar{\beta}_i (i = 0, 1, \dots, k)$ which are coefficients, consist of real numbers and $\bar{\alpha}_{k-m} \neq 0$.
- B. The polynomials of the form:

$$\bar{\rho}(\lambda) \equiv \sum_{i=0}^{k-m} \bar{\alpha}_i \lambda^i; \quad \bar{\delta}(\lambda) \equiv \sum_{i=0}^k \bar{\beta}_i \lambda^i.$$

Do not have a common factor different from the constant.

- C. $\bar{\delta}(1) \neq 0$ also $p \geq 1$ (p – is degree for (14)).

V. Ibrahimov fundamentally studied the method (14), established a connection between order and degree for this method, and proved several theorems. One of these theorems is as follows:

Theorem (Ibrahimov). If method (14) is stable and has the degree p , then the following inequality is true:

$$p \leq k + m + 1 \quad (k \geq 3m).$$

By simple comparison, we can verify that within the class of advanced corresponding methods, there are robust methods that are more accurate than the stable methods of the multistep type. From here, it follows that the advanced methods are

promising. Obviously, one of the ways to construct more accurate and stable methods involves utilizing the terms of the Taylor expansion.

From the previous assumptions, it follows that one of the goals of this study is to compile innovative methods. Typically, MMs are considered innovative if the method is stable and has the highest maximum degree. But here we have added many properties to determine innovative methods, such as the amount of computational work and the stability region for numerical methods. Therefore, in the development of innovative methods for numerical MMs, certain properties are taken into account, including stability, simple structures, maximum degree, and an extended stability region. So, the definition of Simpson’s method is not innovative; it determines when the boundary of the stability region equals zero. Therefore, this section is dedicated to presenting numerical results obtained using simple methods.

Note that the next section is devoted to the study of multistep second-order derivative methods.

3 CONSTRUCTION MULTISTEP ADVANCED METHODS WITH THE SECOND DERIVATIVE

Let us be given a multi-step derivative method up to the second order of the form.

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h \sum_{i=0}^k \beta_i y'_{n+i} + h^2 \sum_{i=0}^s \gamma_i y''_{n+i}. \tag{15}$$

At $m = 0$ and $s = k$, this method is considered in various works (see, for example, [30]–[56]) and was fundamentally studied by Dahlquist in specific cases, $m = 0$) but in its general form $m \neq 0$ and $s \neq k$ fundamentally investigated by V.R. Ibrahimov.

Note that applying method (15) is not an easy task. The question arises about what the connection is with the applications of the values, $y''(x)$. It is easy to define that if, $\delta_1 \neq 0$, then the calculation of $y''(x)$ at the mesh points x_{n+k} will present some difficulties. If we find the second derivative of equation (1), we obtain a complex integral, the calculation of which is more challenging than solving problem (1). To make sure, let us consider:

$$y''(x) = g(x, y) + \delta_1 K(\delta_2 x, x, y(x)) + \delta_1 \int_{x_0}^x K'_x(\delta_2 x, s, y(s)) ds, \quad y(x_0) = y_0, \tag{16}$$

$$y'(x_0) = f(x_0, y_0), \quad x_0 \leq x \leq X,$$

here

$$g(x, y) = f'_x(x, y) + f'_y(x, y)(f(x, y) + \delta_1 \int_{x_0}^x K(x, s, y(s)) ds).$$

In the case of $\delta_1 = 0$, from (16) we obtain the following initial problem for a second-order ODE. It is easy to verify that applying method (15) to solve problem (1) is not difficult for all values m, s, k of the coefficients α_p, β_i and γ_i . Noted that in this case $\delta_1 = 0$ from the problem (16) it follows:

$$y''(x) = g(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = f(x_0, y_0). \tag{17}$$

This problem has been investigated by many scientists such as Shtömer-Verlet and others. An effective method for solving this problem is usually considered to be the following:

$$\sum_{i=0}^k \alpha_i y_{n+i} = h^2 \sum_{i=0}^k \gamma_i g_{n+i} \quad (n = 0, 1, \dots, N - k). \tag{18}$$

This is the generalization of the Shtömer-Verlet method.

Let method (15) converge, and ensure that the following conditions are met for its coefficients:

- A. The coefficients, $\alpha_i (i = 0, 1, \dots, k - m)$, $\beta_i (i = 0, 1, \dots, k)$ and $\gamma_i (i = 0, 1, \dots, s)$ are real numbers and $\alpha_{k-m} \neq 0$.
- B. The polynomials

$$\bar{\rho}(\lambda) \equiv \sum_{i=0}^{k-m} \alpha_i \lambda^i; \quad \bar{\delta}(\lambda) \equiv \sum_{i=0}^k \beta_i \lambda^i, \quad \bar{\gamma}(\lambda) \equiv \sum_{i=0}^s \gamma_i \lambda^i.$$

The terms do not have a common factor other than a constant.

- C. We make sure that conditions $\delta(1) \neq 0$ and $p \geq 1$ are satisfied.

The concept of method stability (15) is determined based on Definition 1. However, the concept of method stability (18) cannot be defined based on Definition 1. We believe that all the methods mentioned earlier should be studied independently.

It is not difficult to see that method (18) can be derived from (15) as a special case. Thus, we are convinced that methods (18) and (15) can be similar. Let us show that this is not the case. Based on the shift operator, $Ey(x) = y(x + h)$, equality (18) is expressed as follows:

$$\rho(E)y_n - h^2\gamma(E)y_n'' = 0. \tag{19}$$

At $h \rightarrow 0$, passing to the limit, we obtain:

$$\rho(1)y(x) = 0, \tag{20}$$

Note that there $x = x_0 + nh$ is a fixed point here. Hence, it follows that $\rho(1) = 0$. Using the Taylor expansion, we can express equation (19) in the form:

$$(\rho(1) + (E - 1) + (E - 1)^2 \rho'(1))y_n - h^2\gamma(E)y_n'' = 0,$$

From here, for the case $h \rightarrow 0$ one can be written:

$$\rho'(E)(y_{n+1} - y_n)/h = h\gamma(E)y_n'' \text{ or } \rho'(E)y_n' = h\gamma'(E)y_n''. \tag{21}$$

If denote $y_n' = Z_n$ and $y_n'' = Z_n'$, then one can be written:

$$\rho'(E)Z_n - h\gamma(E)Z_n' = 0. \tag{22}$$

By comparing this with method (11), we conclude that certain conditions $\rho'(1) = 0$ must be met. Hence, it follows that $\lambda = 1$ the double root is the characteristic polynomial $\rho(\lambda)$ of method (18). From here, it can be proven that $\rho'(1) \neq 0$. Due to the fact that $\lambda = 1$ is a double root of the $\rho(\lambda)$, it is a necessary condition for the convergence of method (18). Thus, we are convinced that methods (15) and (18) have different properties. Using this, Dahlquist defined the concept of the degree of method (18) as follows:

Definition (Dahlquist). We will refer to the integer variable as p the degree of the method (18) if the asymptotic condition holds true.

$$\sum_{i=0}^k (\alpha_i y(x + ih) - h^2 \beta_i y''(x + ih)) = O(h^{p+2}), \quad h \rightarrow 0. \tag{23}$$

In his research, Dahlquist showed that if method (18) is stable and has degree p , then $p \leq 2[k/2] + 2$ is true, and there are also stable methods of degree $p_{max} = 2[k/2] + 2$ for any k .

As noted earlier, if $\delta_1 = 0$ we receive the following from (1):

$$y'(x) = f(x, y), \quad y(x_0) = y_0, \quad x_0 \leq x \leq X, \tag{24}$$

This has been investigated by many authors. Therefore, let's consider a problem of this type:

$$y'(x) = f(x, y) + \int_{x_0}^x \varphi(s, y(s)) ds, \quad y(x_0) = y_0, \quad x_0 \leq x \leq X, \tag{25}$$

which is the partial case the problem (1).

From here, one can start writing:

$$\begin{aligned} y''(x) &= f'_x(x, y) + f'_y(x, y)y'(x) + \varphi(x, y(x)), \\ y(x_0) &= y_0, y'(x_0) = f(x_0, y_0), \quad x_0 \leq x \leq X. \end{aligned} \tag{26}$$

By performing a simple check, you can ensure that when applying method (15) to solve problem (26) using problem (25), no difficulties will arise. We also note that no difficulties arise when studying problem (25) because the function written under the integral is independent of the boundary of the integral. The methods used to solve problem (24) can be easily adapted to solve problem (25).

If the above-described method is applied to solve equation (25), the following result is obtained.

$$\sum_{i=0}^k \alpha_i y_{n+i} = h \sum_{i=0}^k \beta_i f_{n+i} + h \sum_{i=0}^k \beta_i \vartheta_{n+i},$$

here $\vartheta(x) = \int_{x_0}^x \varphi(s, y(s)) ds$, is considered equivalent to the following expression:

$$\vartheta'(x) = \varphi(x, y(s)), \quad \vartheta(x_0) = 0.$$

Here, we are convinced that the methods proposed above can solve problems (25) and (26) with higher-order accuracy to achieve the expected results.

4 NUMERICAL RESULTS

It is noted that the application of numerical methods to solving the IVPs for VIDE, where the first and second derivatives of the solution are utilized, is considered.

By using that, we have studied some MMs with the first and second derivatives and applied them to solve the following problems:

1. $y'' = \lambda^2 + \lambda^3 \int_0^x y(s)ds, y(0) = 1, y'(0) = \lambda, x \in [0, 1],$
2. $y'' = \lambda^2(1 + a(1 - y(x))) + (1 + a)\lambda^3 \int_0^x y(s)ds, y(0) = 1, y'(0) = \lambda, x \in [0, 1],$
3. $y' = \lambda y, y(0) = 1, 0 \leq x \leq 1.$

Note that we can present the exact solution to all the problems under consideration in the form of a formula $y(x) = \exp(\lambda x)$.

To solve problems 1 and 2, let us apply the following method:

$$y_{n+2} = 2y_{n+1} - y_n + h^2 (y''_{n+2} + 10y''_{n+1} + y''_n)/12, \tag{27}$$

Remember the Shtörmer method. The algorithm for applying this method to solve problems 1 and 2 can be constructed by following these steps:

$$\begin{aligned} \hat{y}_{n+2} &= 2y_{n+1} - y_n + h^2 f_{n+1} + h^2 g_{n+1}, \\ g_{n+2} &= g_n + h \left(K(x_{n+2}, x_{n+2}, \hat{y}_{n+2}) + 2K(x_{n+1}, x_{n+1}, y_{n+1}) + 2K(x_{n+2}, x_{n+1}, y_{n+1}) \right. \\ &\quad \left. + 2K(x_n, x_n, y_n) - K(x_{n+1}, x_n, y_n) \right) / 3, \\ y_{n+2} &= 2y_{n+1} - y_n + h^2 \left(f(x_{n+2}, \hat{y}_{n+2}) + 10f(x_{n+1}, y_{n+1}) + f(x_n, x_n) \right) / 12 \\ &\quad + h^2 (g_{n+2} + 10g_{n+1} + g_n) / 12, \quad g(x) = \int_0^x y(s)ds. \end{aligned}$$

The results obtained by applying method (27) to solve problems 1 and 2 are summarized in Tables 1 and 2.

Table 1. Error for method (27) at $\lambda = 1$

Variable x	Step Size $h = 0,1$	Step Size $h = 0,05$	Step Size $h = 0,01$
0.20	4.45E-09	3.8E-10	7.32E-13
0.60	5.29E-08	3.45E-09	5.68E-12
1.00	1.2E-07	7.51E-09	1.2E-11

The results tabulated in Table 1 correspond theoretically. The errors encountered while solving problem 2 using method (27) have been tabulated in Table 2.

Table 2. Error for method (27) at $a = 1; h = 0,01; \lambda = \pm 1i \pm 5$

Variable x	$\lambda = 1$	$\lambda = -1$	$\lambda = 5$	$\lambda = -5$
0.20	2.04E-12	2.16E-12	2.53E-08	3.26E-08
0.60	1.7E-11	2.05E-11	1.17E-08	1.17E-08
1.00	4.1E-11	5.53E-11	2.86E-06	1.97E-07

The results correspond theoretically. Noted that these problems can be considered the same issue. Using the mentioned method (27) to solve the above-mentioned problems is not innovative. Therefore, for the innovative construction method, let us consider the following equations:

$$\begin{aligned} \hat{y}(x_{n+2}) &= y(x_n) + 2hy'(x_{n+1}) + h^3 y'''(x_n)/3 + h^4 y^{IV}(x_n)/3 + O(h)^5, \\ \bar{y}(x_{n+2}) &= y(x_n) + 2h(\hat{y}'(x_{n+2}) + y'(x_{n+1})) + h^3 y'''(x_n)/3 + O(h)^5, \\ y(x_{n+2}) &= (\hat{y}(x_{n+2}) + \bar{y}(x_{n+2}))/2. \end{aligned} \tag{28}$$

Since the third example for an ODE involves an initial value, value problem, in problem (1) we take $\delta_1 = 0$. In this case, for problem (1) we will find approximate solutions. Simpson’s method can be presented as follows:

$$y_{n+2} = (\hat{y}_{n+2} + \bar{y}_{n+2})/2. \tag{29}$$

Now let’s look at the view midpoint method:

$$y_{n+2} = y_{n+1} + hy'(x_{n+1} + h/2),$$

Here, the local truncation error is expressed as: $0(h^3)$. Note that this method differs from others in that it involves the use of a fractional step size. In this case, the fractional step size method can be represented as follows:

$$y_{n+2} = y_n + h(y'(x_{n+1/2-\alpha}) + y'(x_{n+1/2+\alpha}))/2, \alpha = \sqrt{3}/6, \tag{30}$$

In this case, the degree is $p = 4$. It is known that Simpson’s method is stable, and its degree is also $p = 4$. We would like to separately note that method (29) is also stable, with a degree of $p = 4$. Here we have three methods with different properties, but all the methods under consideration have equal degrees. Obviously, one of the popular methods with degree $p = 4$ is Simpson’s method, which, in some sense, coincides with method (29).

Let us apply these methods to solve examples 1–3. By defining the error received for the aforementioned methods as ϵ_1 (for (30)), ϵ_2 (for (29)) and ϵ_3 for Simpson’s method, we have tabulated the corresponding errors. Receiving results, let us tabulate the data in Tables 3 and 4 for examples 1 and 2.

Table 3. Calculations for step size $h = 0.1$

x	$\lambda = 5$			$\lambda = -5$		
	ϵ_1	ϵ_2	ϵ_3	ϵ_1	ϵ_2	ϵ_3
0.1	2.8E-3	1.1E-2	1.1E-2	2.3E-3	4.4E-3	4.4E-3
0.4	5.1E-2	1.4E-1	1.4E-1	2E-3	8.7E-4	8.7E-4
0.7	4E-1	1E-0	1E-0	8.1E-4	3.9E-3	3.9E-3
1	2.5E-0	6.7E-0	6.7E-0	2.5E-4	4.3E-3	4.3E-3

Table 4. Application methods (27), (29) and (30) to solve problem 3

x	$\lambda = 1$			$\lambda = 5$		
	ϵ_1	ϵ_2	ϵ_3	ϵ_1	ϵ_2	ϵ_3
0.1	4.9E-7	5.8E-7	5.8E-7	2.4E-4	1.6E-4	1.6E-4
0.4	1.4E-6	1.7E-6	1.7E-6	2.1E-4	3E-5	3E-5
0.7	1.8E-6	2.2E-6	2.2E-6	8.4E-5	1.9E-4	1.9E-4
1	1.9E-6	2.3E-6	2.3E-6	2.6E-5	3.9E-4	3.9E-4

Note that method (29) and Simpson's method are one and the same. Method (29) consists of several formulas. Given that the local errors of these methods are summed up, one can infer that the error of method (29) should be greater than that of the Simpson method. In our example, this did not happen because the stability region for method (29) and Simpson is the same when using h . If we increase the step-size value, then the difference in error will become noticeable.

Thus, in our case, methods (30) and (29) can be considered innovative, with method (28) being explicit.

5 CONCLUSION

As noted above, this study investigates the construction and application of an innovative method for solving IVPs for the VIDE. For the sake of objectivity, it is important to note that there is a direct connection between the solutions of the IVPs for both the ODE and the VIDE. Based on these relationships, scientists modified the known methods used in solving ODEs and considered the possibility of using them to solve VIDE. In addition, some certifications are provided here in a concise form to identify innovative methods. For this purpose, the utilization of established multi-stage methods and their adaptations in the form of advanced methods and second derivative methods is being considered. Note that by using the given information, one can compare various modifications of MSDM that have been investigated here. We hope that the methods that have been investigated here will find their followers. Multistep methods with constant coefficients are well-known for solving initial-value problems for both ODEs and Volterra integral-differential equations. Here, I have demonstrated that one can develop innovative approaches to address these issues by creating simple yet effective methods. Among these methods, it is important to note that one-step methods have some advantages. The advantages of these methods have been illustrated using a simple model problem.

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7 CONFLICT OF INTEREST

The authors express that there is no minusdestanting between them.

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